> **Info**
>
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$\lim_{x\to a}\frac{f(x)}{g(x)}$ of functions $f,g:U\subseteq\mathbb{R}\to\mathbb{R}$

For a differentiable function $f:(-a,a)\to\mathbb{R}$ we know

$$f(0)=0\implies\lim_{x\to 0}\frac{f(x)}{x}=f'(0)$$

by definition.

> **(L'Hopital) Let $f,g:(a,b)\to\mathbb{R}$ be differentiable, $g,g'\neq 0$ and $\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$. If**
>
> $$\begin{cases}\lim_{x\to a}f(x)=0=\lim_{x\to a}g(x)\\ \text{or}\\ \lim_{x\to a}g(x)=\infty\end{cases}\implies\lim_{x\to a}\frac{f(x)}{g(x)}=A$$

Let $f,g:(a,b)\to\mathbb{R}$ be differentiable, $g,g'\neq 0$.

- Let $\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$
  - we have $\forall\epsilon>0,\exists c_\epsilon$ such that

    $$\theta\in(a,c_\epsilon)\implies\left|\frac{f'(\theta)}{g'(\theta)}-A\right|<\frac{\epsilon}{2}$$

  - For any $(x,y)\subset(a,c_\epsilon)$, then by Ratio MVT, $\exists\theta\in(x,y)$ such that

    $$\frac{f'(\theta)}{g'(\theta)}=\frac{f(x)-f(y)}{g(x)-g(y)}$$
    $$\implies\left|\frac{f'(\theta)}{g'(\theta)}-A\right|=\left|\frac{f(x)-f(y)}{g(x)-g(y)}-A\right|<\frac{\epsilon}{2}$$

  - Hence,$\forall\epsilon,\exists c_\epsilon$ such that

    $$(x,y)\subset(a,c_\epsilon)\implies\left|\frac{f(x)-f(y)}{g(x)-g(y)}-A\right|<\frac{\epsilon}{2}$$

- If $\lim_{x\to a}f(x)=0=\lim_{x\to a}g(x)$, we have

- $\forall\epsilon\ \exists\delta^1_\epsilon$

$$x \in (a, a+\delta^1_\epsilon) \implies \left|\left(\frac{f(x)-f(y)}{g(x)-g(y)} - A\right) - \left(\frac{f(y)}{g(y)} - A\right)\right| < \frac{\epsilon}{2}$$

- $\forall\epsilon$ let $\delta_\epsilon := \min(\delta^1_\epsilon, c_\epsilon)$,

$$|y-a| < \delta_\epsilon$$

- for any $x < y$ we have $(x, y) \subset (a, c_\epsilon)$ and $|x-a| < \delta_\epsilon < \delta^1_\epsilon$ we have both the previous inequalities so

$$\left|\frac{f(y)}{g(y)} - A\right| < \epsilon$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - R1
      - limit of fractions $\lim_{x\to a}\frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \to \mathbb{R}$

And it has 5 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - R1
      - discont types Types of discontinuities of functions on $\mathbb{R}$
      - extremum Extremum of functions $\mathbb{R} \to \mathbb{R}$
      - fixed pts Fixed points of functions $U \subseteq \mathbb{R} \to U$
      - limit of fractions $\lim_{x\to a}\frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \to \mathbb{R}$
      - Var and sequence of functions Total variation and sequence of continuous functions