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Discrete subgroups of $\mathbb{R}^n$

> **Let $\Gamma \leq \mathbb{R}^n$ be a subgroup and $V := \mathbb{R}\langle\Gamma\rangle$ be its $\mathbb{R}$-span. Then Γ is discrete $\iff$ there is a $\mathbb{R}$-linear isomorphism**
>
> $$T : \mathbb{R}^{\dim V} \cong_{\mathsf{Vec}_{\mathbb{R}}} V$$
>
> **such that $T : \mathbb{Z}^{\dim V} \cong_{\mathsf{Ab}} \Gamma$.**

Let $\Gamma \leq \mathbb{R}^n$ be a discrete subgroup and $V := \mathbb{R}\langle\Gamma\rangle$ be its $\mathbb{R}$-span with $\dim V =: k$. Consider a $\mathbb{R}$-basis $\{\gamma_1, \ldots, \gamma_k\}$ of V.

- Let $\Gamma_0 := \mathbb{Z}\langle\gamma_1, \ldots, \gamma_k\rangle \leq \Gamma$.
- $$\begin{array}{ccc} V & \to & \underbrace{V/\Gamma_0}_{\text{compact}} \\ & \searrow & \downarrow \\ & & V/\Gamma \end{array}$$

  is a commutative diagram of covering maps with V/Γ_0 being homeomorphic to $\mathbb{R}^k/\mathbb{Z}^k$ which is compact.
- Thus

  $$\begin{array}{ccc} \underbrace{\Gamma/\Gamma_0}_{\text{finite}} & \curvearrowright & \underbrace{V/\Gamma_0}_{\text{compact}} \\ & & \downarrow \text{finite degree} \\ & & V/\Gamma \end{array}$$

  we conclude Γ/Γ_0 is finite.
- Therefore, Γ is finitely generated, tortion free Abelian group with a finite index subgroup Γ_0 of rank k. By

Let G be a free Abelian group with rank n and $H \leq G$ be a subgroup. Then H is a free Abelian group with rank $n \iff H \leq G$ is of finite index.

tensoring with $\mathbb{Q}$

We have a short exact sequence

$$0 \hookrightarrow H \to G \to {}^{G}\!/_{H} \to 0$$

which when tensored with $\mathbb{Q}$ gives us

$$0 \to \mathbb{Q}^{\operatorname{rank}(H)} \to \mathbb{Q}^{\operatorname{rank}(G)} \to {}^{G}\!/_{H} \otimes \mathbb{Q}$$

- Let $\operatorname{rank}(H) = \operatorname{rank}(G) = n$ implies

  $${}^{G}\!/_{H} \otimes \mathbb{Q} = 0$$

  thus

  $$\operatorname{rank}\left({}^{G}\!/_{H}\right) = \dim_{\mathbb{Q}} {}^{G}\!/_{H} \otimes \mathbb{Q} = 0$$

- Let ${}^{G}\!/_{H}$ be finite. Then

  $${}^{G}\!/_{H} \otimes \mathbb{Q} = 0$$

  thus

  $$0 \to \mathbb{Q}^{\operatorname{rank}(H)} \to \mathbb{Q}^{\operatorname{rank}(G)} \to 0$$

  implying $\operatorname{rank}(H) = \operatorname{rank}(G) = n$.

we conclude Γ also has rank $k = \dim V$.

- Thus

  $$\Gamma = \mathbb{Z}\langle \alpha_1, \ldots, \alpha_k \rangle$$

  such that

  $$V \supseteq \mathbb{R}\langle \alpha_1, \ldots, \alpha_k \rangle \supseteq \mathbb{R}\langle \gamma_1, \ldots, \gamma_n \rangle = V$$

- We construct a linear isomorphism

  $$\begin{aligned} \mathbb{R}^{\dim V} &\cong V \\ e_i &\mapsto \alpha_i \end{aligned}$$

  which takes $\mathbb{Z}^{\dim V}$ to Γ.

Current note has 1 direct children and 1 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - R n discrete subg Discrete subgroups of $\mathbb{R}^n$
      - power sum Power series sum of a discrete subgroup of $\mathbb{R}^2$

And it has 36 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - circle packing Circle packing on $\mathbb{R}^2$
    - circle packing to Riemann map Circle packing converges to the Riemann biholomorphism
    - Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
    - Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$
    - cube dyadic Dyadic cubes
    - curves Curves
    - derivative Differentiable functions
    - forms Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner Fourier-Wigner transform
    - harmonic composed conformal Harmonic functions composed with conformal maps
    - Hilbert Hilbert transform
    - hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Hol sets Holomorphic subsets of $\mathbb{C}^n$
    - hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
    - hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
    - KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

    - Laplace Laplace operator on $\mathbb{R}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - met density Metric density of subsets of $\mathbb{R}^n$

- Mobius n-sphere Mobius endomorphisms
- monotone Monotone functions on $\mathbb{R}$
- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$
- Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$