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Power series sum of a rank 2 discrete subgroup of $\mathbb{R}^2$

https://www.desmos.com/calculator/uj9k5mphsn

Proposition: For a rank 2 discrete subgroup $\Lambda \leq \mathbb{R}^2$ the series converges

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^{2+\delta}} < \infty$$

for $\delta > 0$.

Let $\Lambda = \mathbb{Z}\langle \lambda_1, \lambda_2 \rangle$ and define the parallelogram

$$P(n) := \{a_1\lambda_1 + a_2\lambda_2 \mid a_1, a_2 \in \mathbb{R} : \max\{|a_1|, |a_2|\} = n\}$$

which satisfies

$$P(n) = nP(1)$$

- Thus

$$|P(n) \cap \Lambda| = n\,|P(1) \cap \Lambda| = 8n$$

- Now because

$$\begin{aligned} S^1 &\to \mathbb{R} \\ (a_1, a_2) &\mapsto |a_1\lambda_1 + a_2\lambda_2| \end{aligned}$$

is continuous, the inf

$$k := \inf\{|a_1\lambda_1 + a_2\lambda_2| \mid a_1^2 + a_2^2 = 1; a_1, a_2 \in \mathbb{R}\}$$

is finite.

- But

$$k = 0 \implies a_1\lambda_1 + a_2\lambda_2 = 0; a_1, a_2 \neq 0$$

  which contradicts linear independence of λ_1, λ_2.
- Thus

$$k > 0$$

- This gives

$$a_1\lambda_1 + a_2\lambda_2 \in P(n) \implies |a_1\lambda_1 + a_2\lambda_2|^2 \geq (a_1^2 + a_2^2)k^2 \geq n^2k^2$$

- Choosing some enumeration of Λ via the increasing collection of finite sets $\Lambda \cap P(n)$ as $n \to \infty$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{|\lambda|^{2+\delta}} &= \sum_{n\geq 1} |\Lambda \cap P(n)| \sum_{\lambda\in P(n)\setminus\{0\}} \frac{1}{|\lambda|^{2+\delta}} \\ &\leq \sum_{n\geq 1} \frac{8n}{n^{2+\delta}k^{2+\delta}} \\ &= \frac{8}{k^{2+\delta}} \sum_{n\geq 1} \frac{1}{n^{1+\delta}} < \infty\end{aligned}$$

  for any $\delta > 0$.

critical power on the moduli of rank 2 discrete subgroups

$$GL(2)(\mathbb{R}) \Big/ GL(2)(\mathbb{Z}) \to \mathbb{R}$$

$$\Lambda \mapsto \inf\left\{ \delta \middle| \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{|\lambda|^{2+\delta}} < \infty \right\}$$

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  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - R n discrete subg Discrete subgroups of $\mathbb{R}^n$
      - power sum Power series sum of a discrete subgroup of $\mathbb{R}^2$

And it has 1 siblings.

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