

Info

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Probability distribution of flow

[1]

$$\begin{aligned}\Phi_F^t(x) &= x + tF(x) + O(t^2) \\ \det \mathcal{D}_x \Phi_F^t &= 1 + t(\operatorname{div} F(x)) + O(t^2)\end{aligned}$$

☰ $F \in \operatorname{Vec}^1(\mathbb{R}^n)$ be a complete vector field, then

$$\left. \frac{d}{dt} \operatorname{vol}_{\mathbb{R}^n}(\Phi_F^t A) \right|_{t=0} = \int_A \operatorname{div} F$$

☰ $F \in \operatorname{Vec}^1(\mathbb{R}^n)$ be a complete vector field, $\operatorname{div} F = 0 \implies \Phi_F^t$ preserves volume,

$$\operatorname{vol}_{\mathbb{R}^n}(\Phi_F^t A) = \operatorname{vol}_{\mathbb{R}^n}(A)$$

Definition. Evolution of a probability distribution by flows of a vector field

Let ρ_0 be a probability distribution on \mathbb{R}^n then define the distribution ρ_t^F evolved by F as that satisfies

$$\int_A \rho_t^F = \int_{\Phi_F^{-t} A} \rho_0$$

for any $A \subset \mathbb{R}^n$.

- $$\int_{y \in A} \rho_t^F(y) = \int_{x \in \Phi_F^{-t} A} \rho_t^F(\Phi_F^t(x)) (\det \mathcal{D}_x \Phi_F^t) = \int_{x \in \Phi_F^{-t} A} \rho_0(x)$$

- As A is arbitrary, the integrands must be equal

$$\iff \rho_t^F(\Phi_F^t(x)) \underbrace{(\det \mathcal{D}_x \Phi_F^t)}_{1+t(\operatorname{div} F(x))+O(t^2)} = \rho_0(x)$$

- $\implies \rho_0 = \rho_t(\Phi^t)(1 + t\operatorname{div} F + O(t^2))$
- $\iff \rho_0 - \rho_t(\Phi^t) = \rho_t(\Phi^t)(t\operatorname{div} F + O(t^2))$

☰ $F \in \operatorname{Vec}^1(\mathbb{R}^n)$ be a complete vector field,

$$\frac{d}{dt} \rho_t \circ \Phi_F^t = \partial_t \rho_t + \partial_F(\rho_t) = -\rho_t \operatorname{div} F$$

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 - [probability distribution](#) Probability distribution of flow

And it has 10 siblings.

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1. [Roderich Tumulka - Lecture Notes on Mathematical Statistical Physics.pdf](#) > page=36 ↩