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Volume change by flows on $\mathbb{R}^n$

Consider a **smooth** flow φ^t on some open subset of $\mathbb{R}^n$, this means

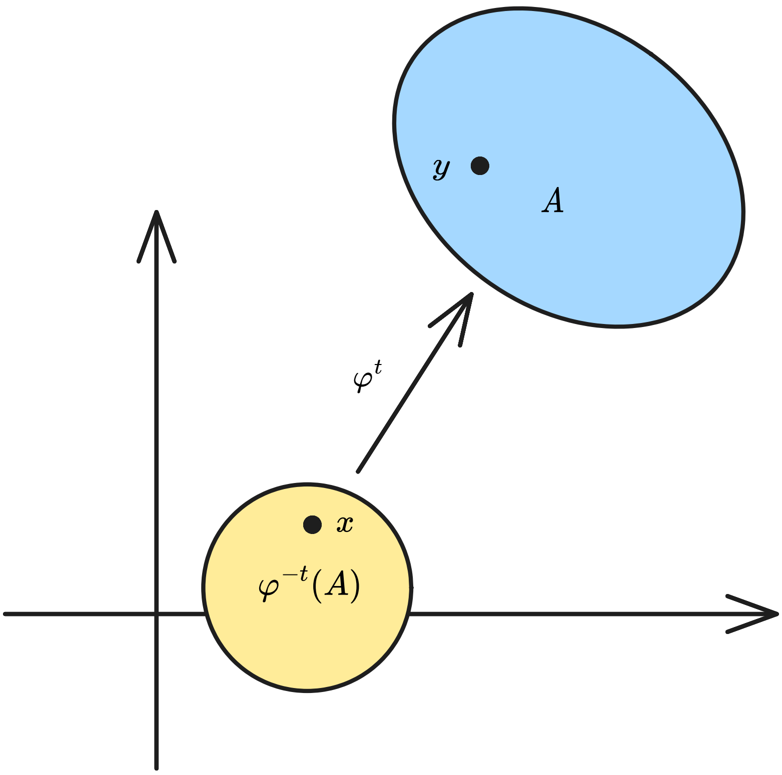
$$\varphi^t : U_t \subseteq U \to \mathbb{R}^n$$

is an injective local diffeomorphism. The *only* problem is surjectivity!

Then choose $A \subseteq \mathbb{R}^n$ so that it's preimage is not empty, then

$$\varphi^t : \varphi^{-t}(A) \to A$$

is a diffeomorphism.



We wish to compare volume of A with volume of $\varphi^{-t}(A)$. Thus we calculate volume

- By *change of variables* we get

$$\begin{aligned}\mu(A) &= \int_{y\in A} \mathrm{d}y_1 \wedge \dots \mathrm{d}y_n \\ &= \int_{x\in\varphi^{-t}(A)} \mathrm{d}\varphi_1^t(x) \wedge \cdots \wedge \mathrm{d}\varphi_n^t(x) \\ &= \int_{x\in\varphi^{-t}(A)} (\det \mathfrak{D}\varphi^t)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\end{aligned}$$

- So

$$\mu(\varphi^{-t}(A)) - \mu(A) = \int_{x\in A} (1 - \det \mathfrak{D}\varphi^t)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

- Now

$$\varphi^0 = \mathsf{Id} \implies \det \mathfrak{D}\varphi^0 = 1$$

  which means

$$\lim_{t\to 0} \frac{1}{t}(1 - \det \mathfrak{D}\varphi^t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det \mathfrak{D}\varphi^t$$

- Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x\in A} \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det \mathfrak{D}\varphi^t\right)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

So rate of change of volume boils down to $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det \mathfrak{D}\varphi^t$.

computing derivative of $\det \mathfrak{D}\varphi^t$ using wedge products

Bug

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathfrak{D}\varphi^t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\varphi^t}{\partial x_1} \wedge \cdots \wedge \frac{\partial\varphi^t}{\partial x_n}\right) \\ &= \frac{\partial}{\partial x_1} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t}_{X(\varphi^t(x))} \wedge \frac{\partial\varphi^t}{\partial x_2} \wedge \cdots \wedge \frac{\partial\varphi^t}{\partial x_n} + \dots \\ &= \left(\mathfrak{D}X\left(\frac{\partial}{\partial x_1}\varphi^t\right)\right)\wedge \\ &= \underbrace{\frac{\partial X_1}{\partial x_1}\det \mathfrak{D}\varphi^t + \dots}_{\det(\mathfrak{D}\varphi^t)\operatorname{div}X}\end{aligned}$$

computing derivative of $\det \mathfrak{D}\varphi^t$ using Jacobi's formula

From

(Jacobi's formula)

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t) = (\det A(t))\mathrm{tr}\left(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)$$

and

$$\begin{aligned} A'(t) &= B(t)A(t) \\ \implies (\det A)'(t) &= \det(A(t))\mathrm{tr}(A(t)^{-1}B(t)A(t)) \\ &= \det(A(t))\mathrm{tr}(B(t)) \end{aligned}$$

along with

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t,x) &= X_{\varphi(t,x)} \\ \implies \frac{\partial}{\partial t}\mathfrak{D}\varphi(t,x) &= \mathfrak{D}X_{\varphi(t,x)} \\ &= \mathfrak{D}_{\varphi(t,x)}X \circ \mathfrak{D}\varphi(t,x) \end{aligned}$$

we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}\varphi^t = \mathfrak{D}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t &= \mathfrak{D}X_{\varphi^t} = \mathfrak{D}X \circ \mathfrak{D}\varphi^t \\ \implies \mathrm{tr}(\mathfrak{D}\varphi^{-t} \circ \mathfrak{D}X \circ \mathfrak{D}\varphi^t) &= \mathrm{tr}(\mathfrak{D}X) \end{aligned}$$

So we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\det\mathfrak{D}\varphi^t_{(t,x)} &= \mathrm{tr}(\mathfrak{D}X)_{(\varphi^t(x))}\det\mathfrak{D}\varphi^t_{(t,x)} \\ &= \mathrm{div}(X)_{(\varphi^t(x))}\det\mathfrak{D}\varphi^t_{(t,x)} \end{aligned}$$

- At $t=0$, $\varphi^0 = \mathsf{Id}$ thus we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\det\mathfrak{D}\varphi^t\right)(x) = \mathrm{div}(X)_x$$

- In general

$$\det\mathfrak{D}\varphi^t_{(t,x)} = \exp\left(\int_{[0,t]}\mathrm{div}(X)_{\varphi^t(x)}\mathrm{d}t\right)$$

[1]

[2]

computing derivative of $\det\mathfrak{D}\varphi^t$ using Liouville's formula

[3]

rate of change of volume is determined by integral of divergence

Hence, we have

For flow

$$\exp(tX) =: \varphi^t$$

of a smooth vector field X on a open subset of $\mathbb{R}^n$, the rate of change of volume of $\phi^{-t}(A), A \subseteq \mathbb{R}^n$ at $t = 0$ is the volume integral of divergence of X, on A

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x \in A} \mathrm{div}(X)_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Thus

- $\mathrm{div}(X) \equiv 0 \iff$ the flow of X is *volume preserving*

using Reynolds transport theorem - Wikipedia

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
      - volume Volume change by flows on $\mathbb{R}^n$

And it has 10 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
      - cons Constant of a flow in $\mathbb{R}^n$
      - Euler method Euler's iterative solution method for first order ODEs
      - existence Existence of integral curves of vector fields in $\mathbb{R}^n$
      - fixed Fixed points
      - flows Flow of a vector field in $\mathbb{R}^n$
      - from graphs Graph $\to$ polynomial ODE
      - grad Gradient flows on $\mathbb{R}^n$

- Hamiltonian Hamiltonian vector fields in $\mathbb{R}^{2n}$ with standard symplectic form
- probability distribution Probability distribution of flow
- volume Volume change by flows on $\mathbb{R}^n$

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1. https://math.stackexchange.com/a/3456775/1290493 ↩
2. https://mathoverflow.net/a/327773 ↩
3. https://math.stackexchange.com/a/312623/1290493 ↩