

Info

This note [found here](#)
as a part of [a collection](#)
is written (completely with human hands) by [Rupadarshi Ray](#),
created on December 13, 2024 3:03:06 PM,
and was last modified on June 12, 2026 7:51:52 PM.

$\mathcal{C}_0[0, 1]$

It does not make sense to talk about boundary values for L^p spaces because they are defined upto equality almost everywhere.

So we consider the subspace of continuous functions on $[0, 1]$

$$\mathcal{C}_0[0, 1] := \{f \in \mathcal{C}[0, 1] \mid f(0) = 0 = f(1)\}$$

- Its $\ker \text{eval}_0 \cap \mathcal{C}[0, 1] \cap \ker \text{eval}_1$.
- $\mathcal{C}[0, 1]_{0,0} \hookrightarrow \mathcal{C}(S^1) \hookrightarrow \mathcal{C}[0, 1]$
- 0 is the only constant function in the domain.

integration

So we consider the subspace of continuous functions on $[0, 1]$

$$\mathcal{C}_0[0, 1] := \{f \in \mathcal{C}[0, 1] \mid f(0) = 0 = f(1)\}$$

$$\int_{[0,1]} : \mathcal{C}[0, 1]_{0,0} \rightarrow \mathbb{C}$$

integral

So we consider the subspace of continuous functions on $[0, 1]$

$$\mathcal{C}_0[0, 1] := \{f \in \mathcal{C}[0, 1] \mid f(0) = 0 = f(1)\}$$

$$\int_{[0,\bullet]} : \mathcal{C}[0, 1]_{0,0} \rightarrow \mathcal{C}[0, 1]_{0,-}$$

- The operator has no eigenvalues as

$$\int_{[0,x]} f = \lambda f$$

$$\implies f = \lambda f', f(0) = 0$$

$$\implies f = 0$$

- The "inverse" operator is derivative

$$\mathcal{D} : \mathcal{C}^1[0, 1]_{0,0} \rightarrow \mathcal{C}[0, 1]$$

$$\left| \int_{[0,x]} f \right| \leq \int_{[0,x]} |f|$$

$$\leq \|f\|_\infty \int_{[0,x]} 1$$

$$\leq \|f\|_\infty$$

$$\left\| \int_{[0,\bullet]} f \right\|_\infty \leq \|f\|_\infty$$

•

derivative

$$\mathcal{D} : \mathcal{C}^1[0, 1]_{0,0} \rightarrow \mathcal{C}[0, 1]$$

double derivative and its domain

Definition. Space of continuously double differentiable functions on $[0, 1]$ with boundary value 0

We consider our domain to be

$$\mathcal{D}(\mathcal{D}^2) := \{f \in \mathcal{C}^2([0, 1], \mathbb{C}) \mid f(0) = 0 = f(1)\}$$

which is a subspace of $L^2[0, 1]$. On this domain we have the operator

$$\mathcal{D}^2 : \mathcal{D}(\mathcal{D}^2) \subset L^2[0, 1] \rightarrow \mathcal{C}[0, 1] \subset L^2[0, 1]$$

- The domain is $\ker \text{eval}_0 \cap \mathcal{C}^2[0, 1] \cap \ker \text{eval}_1$.
- $0 \in \mathcal{D}^2(\mathcal{D})$ is the only constant function in this domain.

Proposition: For

Definition. Space of continuously double differentiable functions on $[0, 1]$ with boundary value 0

We consider our domain to be

$$\mathcal{D}(\mathfrak{D}^2) := \{f \in \mathcal{C}^2([0,1], \mathbb{C}) \mid f(0) = 0 = f(1)\}$$

which is a subspace of $L^2[0,1]$. On this domain we have the operator

$$\mathfrak{D}^2 : \mathcal{D}(\mathfrak{D}^2) \subset L^2[0,1] \rightarrow \mathcal{C}[0,1] \subset L^2[0,1]$$

$\mathcal{D}(\mathfrak{D}^2)$ is dense in $L^2[0,1]$. \mathfrak{D}^2 is a linear but not a bounded operator on $\mathcal{D}(\mathfrak{D}^2)$.

Proposition: The set of functions

$$\{\sin(n\pi x) \mid n \in \mathbb{Z}_{>0}\}$$

a set of orthogonal eigenbasis of $L^2[0,1]$. Moreover, they are an orthogonal eigenbasis for \mathfrak{D}^2 on $(\mathcal{D}(\mathfrak{D}^2), \|\cdot\|_2)$.

```
top= 1.2
bottom= -1.2
left=-0.7
right=1.7
---
\sin(\pi x) | 0<x<1
\sin(2\pi x) | 0<x<1
```

- For $n \in \mathbb{Z}_{>0}$

$$\|\sin n\pi x\|_2^2 = \int_{[0,1]} (\sin(n\pi x))^2 = \frac{1}{2}$$

- However, this basis is not an eigenbasis of \mathfrak{D} as

$$\mathfrak{D}(\sin n\pi x) = (n\pi) \cos(n\pi x)$$

- In this basis,

$$\mathfrak{D}^2 = \begin{bmatrix} -\pi^2 & & & & \\ & -4\pi^2 & & & \\ & & \ddots & & \\ & & & -(k\pi)^2 & \\ & & & & \ddots \end{bmatrix}$$

which easily seen to be not a bounded operator.

exponentiating the double derivative

? Question

We want to solve the initial and boundary value problem on $[0, 1]$ of the **Heat equation**

- Solve for a function $u(x, t)$ (regular enough)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \text{ on } (x, t) \in [0, 1] \times (0, \infty)$$

- for given *initial value*

$$x \in \mathbb{R} \implies u(x, 0) = f(x)$$

for some $f \in L^2[0, 1]$

- and *boundary conditions*

$$t \in \mathbb{R} \implies u(0, t) = 0 = u(1, t)$$

From

- In this basis,

$$\mathcal{D}^2 = \begin{bmatrix} -\pi^2 & & & & \\ & -4\pi^2 & & & \\ & & \ddots & & \\ & & & -(k\pi)^2 & \\ & & & & \ddots \end{bmatrix}$$

which easily seen to be not a bounded operator.

we can define the exponential

$$\exp(t\mathcal{D}^2) := \begin{bmatrix} e^{-t(\pi^2)} & & & & \\ & e^{-t(4\pi^2)} & & & \\ & & \ddots & & \\ & & & e^{-t(k\pi)^2} & \\ & & & & \ddots \end{bmatrix}$$

that is

Definition. Exponential of \mathcal{D}^2

For any $t \in (0, \infty)$ the exponential of \mathcal{D}^2 on its domain with boundary values 0

$$\begin{aligned} \exp(t\mathcal{D}^2) : \mathcal{D}(\mathcal{D}^2) &\rightarrow \mathcal{D}(\mathcal{D}^2) \\ \exp(t\mathcal{D}^2) \left(\sum_{n \in \mathbb{Z}_{>0}} a_n \sin n\pi x \right) &:= \sum_{n \in \mathbb{Z}_{>0}} a_n e^{-t(n\pi)^2} \sin(n\pi x) \end{aligned}$$



◀ **Definition. Exponential of \mathcal{D}^2**

For any $t \in (0, \infty)$ the exponential of \mathcal{D}^2 on its domain with boundary values 0

$$\begin{aligned} \exp(t\mathcal{D}^2) : \mathcal{D}(\mathcal{D}^2) &\rightarrow \mathcal{D}(\mathcal{D}^2) \\ \exp(t\mathcal{D}^2) \left(\sum_{n \in \mathbb{Z}_{>0}} a_n \sin n\pi x \right) &:= \sum_{n \in \mathbb{Z}_{>0}} a_n e^{-t(n\pi)^2} \sin(n\pi x) \end{aligned}$$

is a bounded operator, thus has an **unique bounded extension**

$$\exp(t\mathcal{D}^2) : L^2[0, 1] \rightarrow L^2[0, 1]$$

as $\mathcal{D}(\mathcal{D}^2) \subset L^2[0, 1]$ is dense and $L^2[0, 1]$ is complete.

Because for each $t > 0$ there is a exponential factor $a_n e^{-(n\pi)^2 t}$ the sine-Fourier series actually uniformly converges to a very nice function!

☀ As $(\sin(n\pi x))/\sqrt{2}$ is a orthonormal basis and let $f \in L^2[0, 1]$

$$f = \sum_{n \geq 0} \sqrt{2} a_n \frac{\sin(n\pi x)}{\sqrt{2}}$$

So (a_n) is a square integrable sequence, so it must be bounded.



For $f \in L^2[0, 1]$, for every $t > 0$

◀ **Definition. Exponential of \mathcal{D}^2**

For any $t \in (0, \infty)$ the exponential of \mathcal{D}^2 on its domain with boundary values 0

$$\begin{aligned} \exp(t\mathcal{D}^2) : \mathcal{D}(\mathcal{D}^2) &\rightarrow \mathcal{D}(\mathcal{D}^2) \\ \exp(t\mathcal{D}^2) \left(\sum_{n \in \mathbb{Z}_{>0}} a_n \sin n\pi x \right) &:= \sum_{n \in \mathbb{Z}_{>0}} a_n e^{-t(n\pi)^2} \sin(n\pi x) \end{aligned}$$

converges uniformly to a **smooth** function! Moreover the exponential **solves** the heat equation

$$\frac{\partial}{\partial t} \exp(t\mathcal{D}^2)(f) = \frac{\partial^2}{\partial x^2} \exp(t\mathcal{D}^2)(f)$$

such that the **initial value** is satisfied

$$\exp(t\mathcal{D}^2) \Big|_{t=0} = \text{Id}$$

and the **boundary conditions** are satisfied

$$\exp(t\mathcal{D}^2)(f)(0) = \exp(t\mathcal{D}^2)(f)(1)$$

$\sum_{n \geq 1} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$ has high regularity for (a_n) with low regularity

Proposition: Let $(a_n)_{n \geq 1}$ be a **bounded** sequence. For every $t > 0$ the series

$$u(t, x) := \sum_{n \geq 1} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

converges uniformly to a **smooth** function

$$u(t, -) \in \mathcal{C}^\infty[0, 1]$$

Moreover,

- $t > 0 \implies u(t, 0) = 0 = u(t, 1)$
- $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $(t, x) \in (0, \infty) \times [0, 1]$

? $u(t, -) \rightarrow 0$ as $t \rightarrow \infty$

- Each term in $u(t, x)$ is bounded by

$$\left| a_n e^{-(n\pi)^2 t} \sin(n\pi x) \right| \leq (\sup_n |a_n|) e^{-(n\pi)^2 t}$$

- As

$$\sum_{n \geq 1} (\sup_n |a_n|) e^{-(n\pi)^2 t} = (\sup_n |a_n|) \sum_{n \geq 1} (e^{-\pi^2 t})^{n^2} \leq (\sup_n |a_n|) \sum_{n \geq 1} (e^{-\pi^2 t})^n < \infty$$

by

(Weierstrass M-test) Let for $k \in \mathbb{N}$, $f_k : U \rightarrow \mathbb{R}$ be a sequence of functions such that

$$x \in X, k \in \mathbb{N} \implies |f_k(x)| \leq M_k$$

and

$$\sum_{k \geq 0} M_k < \infty$$

Then the series

$$\sum_{k \geq 0} f_k$$

converges absolutely and uniformly on U .

💡 Let $M_k > 0$ be a sequence of real numbers and f_k be a sequence of functions on $U \subseteq \mathbb{R}$

$$x \in U \implies |f_k(x)| \leq M_k$$

- For $n \geq m$

$$\left| \sum_{k=0}^n f_k(x) - \sum_{k=0}^m f_k(x) \right| = \left| \sum_{k=m+1}^n f_k(x) \right| \leq \sum_{k=m+1}^n M_k$$

- If

$$\sum_{k \geq 0} M_k < \infty$$

then for any $\epsilon > 0$ we have a N_ϵ such that

$$n \geq m > N_\epsilon \iff \sum_{k=m+1}^n M_k < \epsilon$$

then this means

$$\left| \sum_{k=0}^n f_k(x) - \sum_{k=0}^m f_k(x) \right| \leq \sum_{k=m+1}^n M_k < \epsilon$$

- Therefore the series (f_k) is **uniformly Cauchy** on U , thus it converges uniformly.

we conclude $u(t, x)$ converges uniformly on $x \in [0, 1]$ for every $t > 0$.

$$\begin{aligned} \left| \sum_{n=N}^M a_n e^{-(n\pi)^2 t} \sin(n\pi x) \right| &\leq \sum_{n=N}^M |a_n| e^{-(n\pi)^2 t} |\sin(n\pi x)| \\ &\leq \sum_{n=N}^M |a_n| \end{aligned}$$

- Formally

$$\frac{\partial u}{\partial t} = \sum_{n \geq 1} (-n^2 \pi^2) a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

- On $t \in (0, \infty)$

- Formally

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \sum_{n \geq 1} (-1)^k (n^2 \pi^2)^k a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

[1]

[2]([https://rupadarshiray.github.io/notes/wiki.\[Ronald_B._Guenther,_John_W._Lee_-_Partial_Differential_Equations_of_Mathematical_Physics_and_Integral_Equations_\(1988\).pdf#page=163&rect=62,26,556,793.pdf\]](https://rupadarshiray.github.io/notes/wiki.[Ronald_B._Guenther,_John_W._Lee_-_Partial_Differential_Equations_of_Mathematical_Physics_and_Integral_Equations_(1988).pdf#page=163&rect=62,26,556,793.pdf]))

representation of the Heisenberg algebra

$$\left[\hat{x}, \frac{d}{dx} \right] \psi = x \frac{d}{dx} \psi - \frac{d}{dx} (x\psi) = -\psi$$

Thus

$$\left[\hat{x}, -i \frac{d}{dx} \right] = i \text{Id}$$

$$\langle I, X, Y \mid [X, Y] = I \rangle \rightarrow \text{End}(L^2[0, 1])$$

$$I \mapsto (-i) \text{Id}$$

$$X \mapsto (-i) \hat{x}$$

$$Y \mapsto (-i) \hat{p} := -\mathfrak{D} = -\frac{d}{dx}$$

Current note has 0 direct children and 0 total descendants.

- [stamp](#) stamp
 - [Rf](#) subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
 - [cont](#) Continuous functions on \mathbb{R}^d
 - [0to1 w bd](#) $\mathcal{C}_0[0, 1]$

And it has 5 siblings.

- [stamp](#) stamp
 - [Rf](#) subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
 - [cont](#) Continuous functions on \mathbb{R}^d
 - [0to1 w bd](#) $\mathcal{C}_0[0, 1]$
 - [1with inf limit is unf](#) Continuous functions on \mathbb{R} with infinite limit are uniformly continuous
 - [interm](#) Intermediate value property of continuous functions on intervals
 - [space](#) $\mathcal{C}(U, X)$
 - [space cpt](#) $\mathcal{C}[0, 1]$

1. ocw.mit.edu/courses/18-303-linear-partial-differential-equations-fall-2006/d11b374a85c3fde55ec971fe587f8a50_heateqni.pdf#page=42.00 ↩
2. Ronald B. Guenther_ John W. Lee - Partial Differential Equations of Mathematical Physics and Integral Equations-Prentice Hall (1988), p.163 ↩