> **Info**
>
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> as a part of a collection
> is written (completely with human hands) by Rupadarshi Ray,
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Continuous functions on $\mathbb{R}$ with infinite limit are uniformly continuous

continuous function on $[a, \infty]$ is uniformly continuous

> **Let**
>
> $$f : [a, \infty) \to \mathbb{R}$$
>
> **be continuous such that**
>
> $$\lim_{x\to\infty} f(x) = f_\infty \in \mathbb{R}$$
>
> **exists. Then f is uniformly continuous.**

Given $\epsilon > 0$, choose M large enough so that

$$x > M \implies |f(x) - f_\infty| < \frac{\epsilon}{2}$$

- On the **compact** interval $[a, N+1]$ the **continuous** f is **uniformly continuous**. So there is a δ_1 such that

  $$x, y \in [a, N+1], |x-y| < \delta_1 \implies |f(x) - f(y)| < \epsilon$$

- Choose

  $$\delta := \min\{\delta_1, 1\}$$

- Now, consider arbitrary $x, y \in [a, \infty)$ such that $|x-y| < \delta$.
  - If $x, y \in [a, N+1]$,
    - On the **compact** interval $[a, N+1]$ the **continuous** f is **uniformly continuous**. So there is a δ_1 such that

      $$x, y \in [a, N+1], |x-y| < \delta_1 \implies |f(x) - f(y)| < \epsilon$$

- If $x > N+1$ then $|x-y| < \delta < 1 \implies y > N$ which means both $x, y \in [N, \infty)$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_\infty| + |f(y) - f_\infty| \\ &\leq \epsilon \end{aligned}$$

the converse is false and thus it is not a necessary condition

Example

$ \sin x$ is uniformly continuous on $ \R$ but infinite limits do not exist

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Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$
      - 1with inf limit is unf Continuous functions on $\mathbb{R}$ with infinite limit are uniformly continuous

And it has 5 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$
      - 0to1 w bd $\mathcal{C}_0[0,1]$
      - 1with inf limit is unf Continuous functions on $\mathbb{R}$ with infinite limit are uniformly continuous
      - interm Intermedia value property of continuous functions on intervals
      - space $\mathcal{C}(U, X)$
      - space cpt $\mathcal{C}[0,1]$