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Intermedia value property of continuous functions on intervals

continuity keeps the same sign locally

An immediate property of continuous functions on $\mathbb{R}$ is that they do not change sign *abruptly* if the domain is an interval.

(Continuity preserves sign locally): Let $I(c)$ be an interval containing c and

$$f : I(c) \subset \mathbb{R} \to \mathbb{R}$$

be continuous at $c \in I(c), f(c) \neq 0$. Then for $\epsilon := |f(c)|/2$ there is an interval around c such that

$$\exists I_\delta(c) \subseteq f^{-1}(I_\epsilon(c)) \subseteq I(c)$$

Thus on this interval, image of f is in

$$I_\epsilon(c) = f(c) + \left(-\frac{|f(c)|}{2}, \frac{|f(c)|}{2}\right)$$

$\implies$ **f has the same sign as $f(c)$ on $I_\delta(c)$.**

More explicitly,

- Continuity at c means for every $\epsilon > 0, \exists\delta(\epsilon) > 0$ such that

$$x \in I_{\delta(\epsilon)}(c) \cap I(c) \implies f(x) \in I_\epsilon(f(c))$$

  - If $f(c) > 0$, take $\epsilon := \frac{f(c)}{2}$ so

$$\begin{aligned} x \in I_{\delta(f(c)/2)} \cap I(c) &\implies f(x) \in \left(f(c) - \frac{f(c)}{2}, f(c) + \frac{f(c)}{2}\right) \\ &\implies f(x) \in \left(\frac{1}{2}f(c), \frac{3}{2}f(c)\right) \end{aligned}$$

- Thus $f(x)$ has same sigh as $f(c)$ in this domain.

- If $f(c) < 0$, take $\epsilon := -\frac{1}{2}f(c)$ so

$$x \in I \cap I(c) \implies f(x) \in \left(f(c) + \frac{f(c)}{2}, f(c) - \frac{f(c)}{2}\right)$$
$$\implies f(x) \in \left(\frac{3}{2}f(c), \frac{1}{2}f(c)\right)$$

- Thus $\exists\delta\text{–}B(c)$ such that $f(x), x \in \delta\text{–}B(c) \cap S$ has same sign as $f(c)$.

continuous maps over intervals does not change sign without being zero

Using the property that continuity keeps the same sign for an interval we can show if a function (on a closed interval) does change sign, then it has to be zero at some point inside the interval.

$f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous and suppose $f(a), f(b)$ have opposite sign, that is,

$$f(a)f(b) < 0$$

Then $\exists c \in (a, b)$ such that

$$f(c) = 0$$

continuous maps over intervals takes every value between two values

But because we can scale up and down any number really, this idea holds for any real, not just zero.

(Intermediate value theorem for continuous $[a, b] \to \mathbb{R}$) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose there are two points $\alpha < \beta$ in $[a, b]$ such that $f(\alpha) \neq f(\beta)$. Then $f(x)$ takes every value between $f(\alpha)$ and $f(\beta)$ for $x \in (\alpha, \beta)$. That is

$$f(\alpha, \beta) \supseteq (f(\alpha), f(\beta))$$

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- stamp stamp

- Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
  - cont Continuous functions on $\mathbb{R}^d$
    - interm Intermedia value property of continuous functions on intervals

And it has 5 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$
      - 0to1 w bd $\mathcal{C}_0[0,1]$
      - 1with inf limit is unf Continuous functions on $\mathbb{R}$ with infinite limit are uniformly continuous
      - interm Intermedia value property of continuous functions on intervals
      - space $\mathcal{C}(U, X)$
      - space cpt $\mathcal{C}[0,1]$