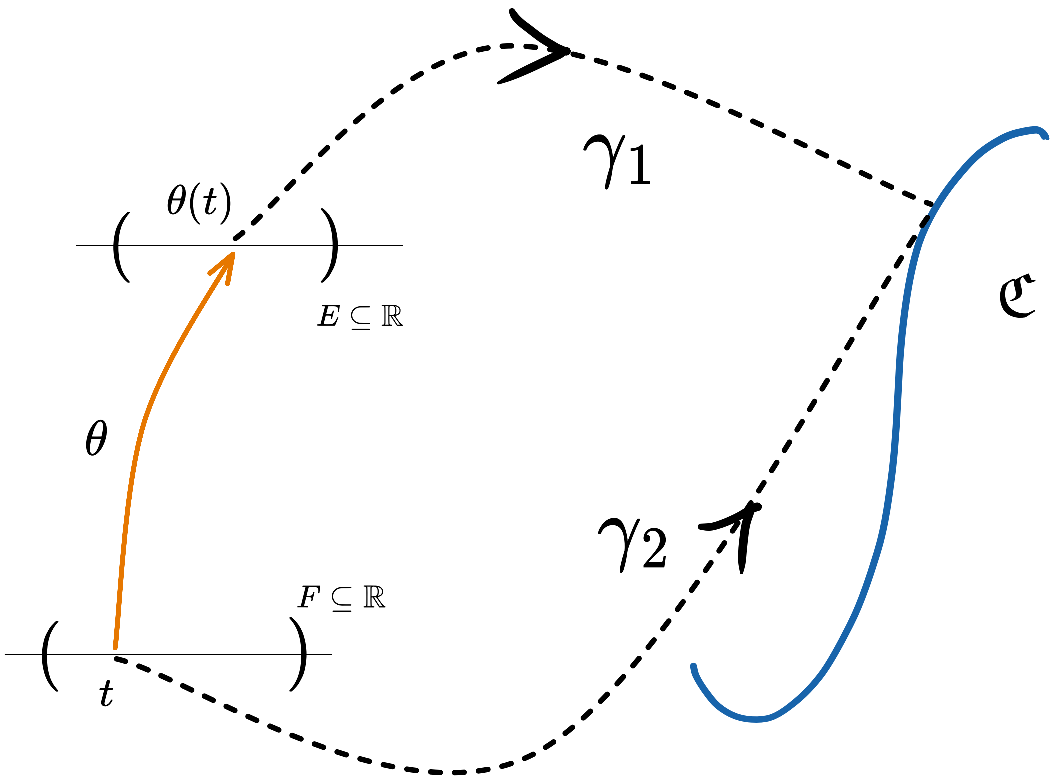
> **Info**
>
> This note found here
> as a part of a collection
> is written (completely with human hands) by Rupadarshi Ray,
> created on September 3, 2022 4:37:05 PM,
> and was last modified on June 12, 2026 11:40:39 AM.

Reparameterization of a *parameterized* curve

Definition. Reparameterizations of *parameterized* curves



Given two parameterized curves $\gamma_1 : E \to \mathbb{R}^n, \gamma_2 : F \to \mathbb{R}^n$, then γ_2 is called a **reparameterization** of γ_1 if there exists a *diffeomorphism* $\theta : F \to E$ such that

$$\gamma_2(t) = (\gamma_1 \circ \theta)(t), \ \forall t \in E$$

This θ is a *"coordinate transformation"* where the previous coordinate was t and the final coordinate is $\theta(t)$.

orientation

Any diffeomorphism θ is strictly monotonic,

$$\theta' > 0 \text{ or } < 0$$

Thus we can define a measurement of the sign of this change.

Definition. Orientation preservation

$$\alpha_O[\theta] := \frac{\theta'}{|\theta'|} = \begin{cases} 1 \text{ if } \theta' > 1 \\ -1 \text{ if } \theta' < 1 \end{cases}$$

If $\alpha_O[\theta] = 1$, then θ is called a **orientation preserving transformation**, and when it is -1 , it is **orientation reversing**.

Obviously, $\alpha_O^2 = 1$ for any θ. This is an important property to check because

$$(\gamma \circ \theta)'(t) = (\gamma' \circ \theta)(t)\, \theta'(t)$$

so, orientation preserving transformations keep the direction of velocity the same, and orientation reversing, reserves the direction of velocity.

Set of all diffeomorphisms of a curve

$$CT[\text{curve } \mathfrak{C}] := \{\theta \mid \theta \text{ is a diffeomorphism of } \mathfrak{C}\}$$

is a group under composition of function

$$(CT[\text{curve } \mathfrak{C}], \circ)$$

and *reparameterization* is a group action on set of all parameterizations of the curve.

The transformations include:

- Ones that preserve speed at every point

$$\theta(t) = \pm t + t_0$$

- Ones that scale velocity by λ

$$\theta(t) = \lambda t$$

parameterization properties

The quantities that change with parameterization in general

- **"velocity"**

- $\gamma' \to (\gamma' \circ \theta)(\theta')$
- **"acceleration"**
  - $\gamma'' \to (\gamma'' \circ \theta)(\theta')^2 + (\gamma' \circ \theta)(\theta'')$

geometric properties

The quantities that change with parameterization in a particular way

- Arc length
- Tangent unit of a curve
  - $$\hat{\mathbf{T}}[\gamma \circ \theta] = \alpha_O[\theta]\ (\hat{\mathbf{T}}[\gamma] \circ \theta)$$

    $$\frac{(\gamma \circ \theta)'}{\|(\gamma \circ \theta)'\|} = \frac{\gamma'(\theta(t))\theta'(t)}{\|\gamma'(\theta(t))\theta'(t)\|} = \alpha_O[\theta]\frac{\gamma' \circ \theta}{\|\gamma' \circ \theta\|}$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - curves Curves
      - reparameterization Reparameterization of a parameterized curve

And it has 6 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - curves Curves
      - arc Arc length
      - congruence Congruent curves
      - curvature Curvature of a curve
      - reparameterization Reparameterization of a parameterized curve
      - tangent Tangent of a curve
      - unit speed Unit speed parameterization of a curve