ⓘ **Info**

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Functions $(a, b) \to \mathbb{R}$ differentiable at a point

Definition. Derivative of functions on $\mathbb{R}$

Let $f : I(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $x \in I$. Then

$$\begin{aligned} I \setminus \{x\} &\to \mathbb{R} \\ t &\mapsto \frac{f(t) - f(x)}{t - x} \end{aligned}$$

is well-defined. Then $f'(x) \in \mathbb{R} \cup \{\infty, -\infty\}$ is defined to be

$$f'(x) := \lim_{(a,b)\setminus\{x\}\ni t \to x} \frac{f(t) - f(x)}{t - x}$$

whenever the right side exists in $\mathbb{R} \cup \{\infty, -\infty\}$.
The function f is called **differentiable** at p if $f'(p) \in \mathbb{R}$.
Higher derivatives are defined iteratively

$$\begin{aligned} f^{(0)} &:= f \\ f^{(n)}(x) &:= (f^{(n-1)})'(x) \end{aligned}$$

whenever they exist.

- By definition, derivative at boundary points ∂I makes sense.
- The function

$$\begin{aligned} (a, b) \times [a, b] \setminus \{t = x\} &\to \mathbb{R} \\ (t, x) &\mapsto \frac{f(t) - f(x)}{t - x} \end{aligned}$$

derivative

- Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.

- Define

$$f_c^*(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

  - f^* is continuous in (a, b).
- So we can write f as

$$f(x) - f(c) = (x-c) f_c^*(x)$$

  for an unique continuous f_c^*.

- Assume can write a function $f : (a, b) \to \mathbb{R}$ as

$$f(x) - f(c) = (x-c) f_c^*(x)$$

  for some continuous f_c^* for some $c \in (a, b)$.
  - Then for any $x \neq c$, taking

$$\frac{f(x)-f(c)}{x-c} = f_c^*(x)$$

    and then the limit $x \to c$, shows $f'(c) = f_c^*(c)$.

Thus, we proved that

> **A function $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ $\iff$ $\exists$ a continuous function $f_c^* : (a, b) \to \mathbb{R}$ such that**
>
> $$f(x) = f(c) + (x-c) f_c^*(x)$$
>
> **for all $x \in (a, b)$. In any case $f'(c) = f_c^*(c)$.**

- From
  - Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.
    - Define

$$f_c^*(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

      - f^* is continuous in (a, b).
    - So we can write f as

$$f(x) - f(c) = (x-c) f_c^*(x)$$

      for an unique continuous f_c^*.

  we write

$$f(x) = f(c) + (x-c) f_c^*(x)$$

as

$$f(x) = f(x) + (x - c)f'(c) + (x - c)h_c(x)$$

where

$$\begin{aligned} h_c(x) &:= f_c^*(x) - f(x) \\ \implies \lim_{x \to c} h_c(x) &= 0 \end{aligned}$$

$f : (a, b) \to \mathbb{R}$ being differentiable at $c \in (a, b)$ $\implies$ f is continuous at c.

non-zero derivative

(Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ and

$$f'(c) > 0 \text{ or } +\infty$$

then there is a $I(c) \subseteq (a, b)$ in which $f(x) - f(c)$ has same sign as $x - c$, that is, f is strictly increasing on $I(c)$.

derivative at extremums

(Derivative, if it exists, must be 0 at local extremas) Let a function $f : (a, b) \to \mathbb{R}$ have a local maxima or minima at $c \in (a, b)$. If f has an extended derivative at c then the derivative must be zero

$$f'(c) \in \mathbb{R} \cup \{-\infty, \infty\} \implies f'(c) = 0$$

Therefore,

$$\{\text{local extremum of } f\} \subseteq Z(f')$$

By

(Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ and

$$f'(c) > 0 \text{ or } +\infty$$

then there is a $I(c) \subseteq (a, b)$ in which $f(x) - f(c)$ has same sign as $x - c$, that is, f is strictly increasing on $I(c)$.

- If $f'(c) > 0$, then by

- Let $f:(a,b)\to\mathbb{R}$ be differentiable at $c\in(a,b)$.
  - Define

$$f_c^*(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x\neq c \\ f'(c) & \text{if } x=c \end{cases}$$

    - f^* is continuous in (a,b).
  - So we can write f as

$$f(x)-f(c)=(x-c)f_c^*(x)$$

  for an unique continuous f_c^*.

we have

$$f(x)-f(c)=(x-c)f_c^*(x)$$

for a continuous f_c^*

- By

(Continuity preserves sign locally): Let $I(c)$ be an interval containing c and

$$f:I(c)\subset\mathbb{R}\to\mathbb{R}$$

be continuous at $c\in I(c), f(c)\neq 0$. Then for $\epsilon:=|f(c)|/2$ there is an interval around c such that

$$\exists I_\delta(c)\subseteq f^{-1}(I_\epsilon(c))\subseteq I(c)$$

Thus on this interval, image of f is in

$$I_\epsilon(c)=f(c)+\left(-\frac{|f(c)|}{2},\frac{|f(c)|}{2}\right)$$

$\implies$ f has the same sign as $f(c)$ on $I_\delta(c)$.

, there is a $I_\epsilon(c)\subseteq(a,b)$ in which $f_c^*(x)$ has same sign as $f^*(c)=f'(c)>0$.

- This means $f(x)-f(c)$ has same sign as $x-c$.

∎

the derivative cannot be positive or negative or $\pm\infty$, otherwise it will be **strictly increasing** locally. So the derivative must be 0

∎

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - derivative Differentiable functions
            - 1 at a point Functions $(a, b) \to \mathbb{R}$ differentiable at a point

And it has 10 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - derivative Differentiable functions
            - 1 at a point Functions $(a, b) \to \mathbb{R}$ differentiable at a point
            - bd Comparing a function and it derivative
            - cont abs Absolutely continuous functions on $[a, b] \leftrightarrow \int_{[a,-]}(L^1[a, b])$
            - dist Distributional derivatives
            - double circle Double derivative/Laplace operator on the circle
            - frac Fractional derivative
            - limit infinite Infinite limit of derivatives
            - space Space of continuous and continuously differentiable functions on $\mathbb{R}$
            - total Derivative of maps $\mathbb{R}^n \to \mathbb{R}^m$
            - zoom Zooming of a map $\mathbb{R}^n \to \mathbb{R}^m$