Info

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Comparing a function and it derivative

Lipshitz bounds and bounds on derivatives

bounded derivative $\iff$ Lipshitz

Let

$$f : U(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$$

be differentiable.

- Then f being M-Lipshitz implies

  $$\left| \frac{f(y) - f(y+h)}{h} \right| \leq M \implies |f'(y)| \leq M$$

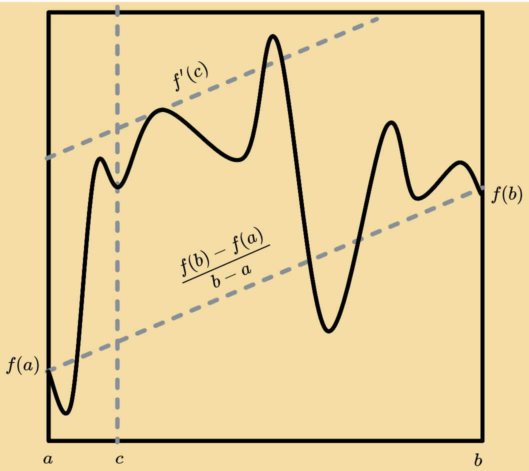
  so f has a bounded derivative on U.
- Let f have a bounded derivative on U, $|f'| \leq M$. Then for any $x, y \in U$, by

  (Lagrange's mean value theorem) Let a continuous function

  $$f : [a, b] \to \mathbb{R}$$

  be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

  $$f(b) - f(a) = (b - a) f'(c)$$



we have

$$\exists c \in (x,y) : \left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \leq M$$

Thus, f is **M-Lipschitz**.

$\mathcal{C}^1 \implies$ locally Lipschitz

Let

$$f : [a,b] \to \mathbb{R}$$

be differentiable and

$$f' : [a,b] \to \mathbb{R}$$

be continuous. Then f' is bounded, say $|f'| \leq M$. Then for any $x, y \in [a,b]$ by

(Lagrange's mean value theorem) Let a continuous function

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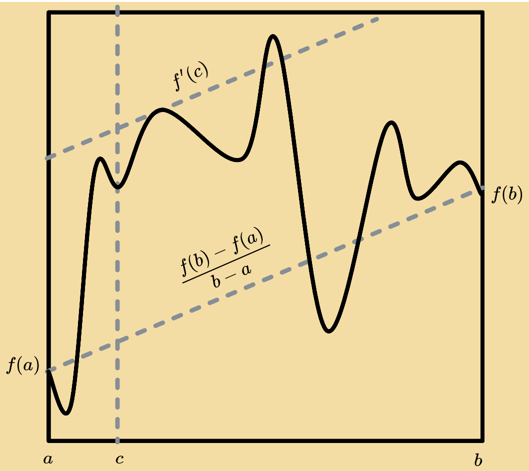
be differentiable on (a,b). Then there is a point $c \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(c)$$



we have

$$\exists c \in (a,b) : \left| \frac{f(y)-f(x)}{y-x} \right| = |f'(c)| \leq M$$

Hence, f is **Lipshitz continuous** on $[a,b]$.

Comparing $\| \ \|_1$

- Let $f \in W^{1,1}(a,b)$. For any $\xi \in (a,b)$ we have

$$f(x) = f(\xi) + \int_\xi^x f'$$

  - Then the derivation of f from $f(\xi)$ is bounded by

$$\begin{aligned} |f(x) - f(\xi)| &= \left| \int_\xi^x f' \right| \\ &= \left| \int_{[x,\xi]} f' \right| \\ &\leq \int_{(a,b)} |f'| \\ &= \|f'\|_1 \end{aligned}$$

  - This implies, in particular, we can bound f as well

$$\begin{aligned} \implies |f(x)| &\leq |f(\xi)| + \int_{(a,b)} |f'| \\ &= |f(\xi)| + \|f'\|_1 \end{aligned}$$

-
  - If $f(a) = 0$ then we have

$$\|f\|_1 \leq \|f'\|_1 |b-a|$$

- **However, if $f(a) \neq 0$, then this inequality does not hold.**

- However, this holds

$$\|f - f(a)\|_1 \leq \|f'\|_1 |b - a|$$

$$\begin{aligned} \mathcal{C}^1(0,1) \setminus \{0\} &\to [0, \infty) \\ f &\mapsto \frac{\|f\|_1}{\|f'\|_1} \end{aligned}$$

Comparing $\| \|_2$

- Let $f \in \mathcal{C}^1(a,b)$ such that $f' \in L^2(a,b)$.
  - Then by

(Intermediate value theorem for continuous $[a,b] \to \mathbb{R}$) Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose there are two points $\alpha < \beta$ in $[a,b]$ such that $f(\alpha) \neq f(\beta)$. Then $f(x)$ takes every value between $f(\alpha)$ and $f(\beta)$ for $x \in (\alpha, \beta)$. That is

$$f(\alpha, \beta) \supseteq (f(\alpha), f(\beta))$$

Let k be a number between $f(\alpha)$ and $f(\beta)$. Then apply

$f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous and suppose $f(a), f(b)$ have opposite sign, that is,

$$f(a)f(b) < 0$$

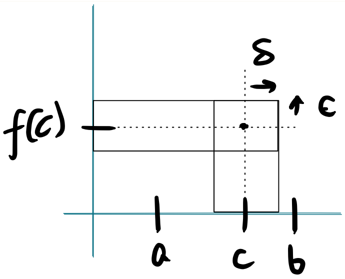
Then $\exists c \in (a,b)$ such that

$$f(c) = 0$$

- Take $f(a) > 0, f(b) < 0$.
  - Define

$$c := \sup\{x \in [a,b] : f(x) \geq 0\}$$

    - This set has a and is bounded above by b. So c exists and $c \in (a,b)$.
  - Assume $f(c) \neq 0$
    - $\exists \delta$ such that δ–$B(c)$ has same sign as $f(c)$ from the sign property
      - If $f(c) > 0$



then

$$x \in (c, \delta) \implies f(x) \geq 0$$

meaning elements of (c, δ) also belong to the above set. **Contradiction** to the definition of c.

- Assume $f(c)[](space. R.1. f. cont.$
- Hence, $f(c) = 0$.

■

for the function $g : [\alpha, \beta] \to \mathbb{R}$

$$g(x) := f(x) - k$$

Use the fact that continuous images of connected sets are connected.

■

we have a $c \in [-r, r]$ such that

$$f(c) = \frac{1}{2r} \int_{(-r,r)} f$$

- Purring $\xi = c$ in
  - Then the derivation of f from $f(\xi)$ is bounded by

$$\begin{aligned} |f(x) - f(\xi)| &= \left| \int_{\xi}^{x} f' \right| \\ &= \left| \int_{[x,\xi]} f' \right| \\ &\leq \int_{(a,b)} |f'| \\ &= \|f'\|_1 \end{aligned}$$

and using Cauchy-Swartz $\|f'\|_1 \leq \left(\int_{(a,b)} 1 \right) \|f'\|_2$ we have

$$|f(x) - f(\xi)| \leq |b - a| \|f'\|_2$$

- ~~Integrating~~

$$\left\| f - \frac{1}{|b-a|}\int_{(a,b)} f \right\|_1 \le |b-a|^2 \|f'\|_2$$

~~which gives us nothing new.~~

- Squaring and integrating, however, produces

$$\left\| f - \frac{1}{|b-a|}\int_{(a,b)} f \right\|_2^2 \le |b-a|^2 \|f'\|_2^2$$

We see

$f \in \mathcal{C}^1[a,b]$	**then**	**and equality** $\iff$
$\int_{[a,b]} f = 0$ and $f(a) = fy(b)$	$\|f\|_2 \le \frac{\|b-a\|}{2\pi}\|f'\|_2$	$f(x) \equiv c\sin\frac{2\pi}{L}(x-a)$ for some $c, \alpha \in \mathbb{R}$
$f(a) = f(b)$	$\|f\|_2 \le \frac{\|b-a\|}{\pi}\|f'\|_2$	$f(x) \equiv c\sin\frac{\pi x}{L}$
$\int_{[a,b]} f = 0$	$\|f\|_2 \le \frac{\|b-a\|}{\pi}\|f'\|_2$	$f(x) \equiv c\cos\frac{\pi x}{L}$

bounding $\|\ \|_p$ on domains with $\mathcal{C}^1$-boundary

(Poincare's inequality) Let $U \subset \mathbb{R}^n$ be a bounded, connected open subset with a $\mathcal{C}^1$-boundary ∂U. For $p \in [1, \infty]$, there exists $c(p, U) > 0$ such that for every $u \in W^{1,p}(U)$ we have

$$\left\| u - \frac{1}{m(u)}\int_U u \right\|_p \le c(p,U)\|\mathfrak{D}u\|_p$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - bd Comparing a function and it derivative

And it has 10 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - 1 at a point Functions $(a, b) \to \mathbb{R}$ differentiable at a point
      - bd Comparing a function and it derivative
      - cont abs Absolutely continuous functions on $[a, b] \leftrightarrow \int_{[a,-]}(L^1[a, b])$
      - dist Distributional derivatives
      - double circle Double derivative/Laplace operator on the circle
      - frac Fractional derivative
      - limit infinite Infinite limit of derivatives
      - space Space of continuous and continuously differentiable functions on $\mathbb{R}$
      - total Derivative of maps $\mathbb{R}^n \to \mathbb{R}^m$
      - zoom Zooming of a map $\mathbb{R}^n \to \mathbb{R}^m$