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Absolutely continuous functions on $[a,b] \leftrightarrow \int_{[a,-]}(L^1[a,b])$

Definition. Absolutely continuous function on $[a,b]$

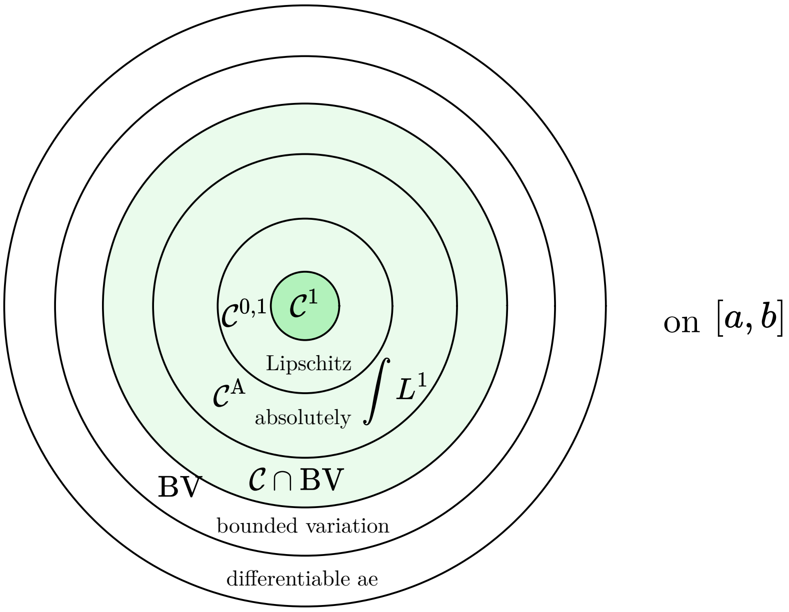
A function F defined on $[a,b]$ is **absolutely continuous** if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that for every collection $(a_k, b_k) \subseteq [a,b]$ of disjoint intervals $1 \leq k \leq N$ we have

$$\sum_{k=1}^{n}(b_k - a_k) < \delta(\epsilon) \implies \sum_{k=1}^{n}|F(b_k) - F(a_k)| < \epsilon$$

The space of all absolutely continuous functions on $[a,b]$

$$\mathcal{C}^{\mathsf{A}}[a,b]$$

may equipped with the supremum $\| \, \|_\infty$ or variation V norms.



$\int_{[a,-]} : L^1[a,b] \to \mathcal{C}^{\mathsf{A}}[a,b]$

- Let $f \in L^1[a,b]$ and consider

$$F(x) := \int_{[a,x]} f$$

  - Then

$$\begin{aligned} F(x) - F(y) &= \int_{[a,x]} f - \int_{[a,y]} f \\ &= \int_{[x,y]} f \\ \implies |F(x) - F(y)| &\le \int_{[x,y]} |f| \end{aligned}$$

  - By

(Absolute continuity of $f dm$) Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every measurable set E such that $m(E) < \delta_\epsilon$

$$\int_E |f| < \epsilon$$

we have $\forall \epsilon > 0 \exists \delta(\epsilon) > 0$ such that for every

$$I = \bigsqcup_k (a_k, b_k)$$

such that

$$m(I) = \sum_k (b_k - a_k) < \delta(\epsilon)$$

the integral

$$\int_I |f| < \epsilon$$

- Here,

$$\begin{aligned} |F(b_k) - F(a_k)| &\leq \int_{[a_k,b_k]} |f| \\ \implies \sum_k |F(b_k) - F(a_k)| &\leq \sum_k \int_{[a_k,b_k]} |f| \\ &= \int_I |f| \\ &< \epsilon \end{aligned}$$

$\mathfrak{D} : \mathcal{C}^{\mathsf{A}}[a,b] \to L^1[a,b]$

(Fundamental theorem of analysis, in the almost everywhere sense) A function

$$F : [a,b] \to \mathbb{R}$$

is *absolutely continuous* on $[a,b]$ $\implies$ F' exists m-almost everywhere and is L^1. In that case,

$$F(x) = F(a) + \int_{[a,x]} F' \qquad \text{on } [a,b]$$

Moreover $F' = 0$ m-almost everywhere $\implies$ F is constant m-almost everywhere.
Conversely, for every $f \in L^1[a,b]$ there exists a function F which is differentiable m-almost everywhere and

$$F' = f \text{ ae}$$

and in fact we may take F to be the absolutely continuous function $x \mapsto c_0 + \int_{[a,x]} f$ for any $c_0 \in \mathbb{C}$.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - cont abs Absolutely continuous functions on $[a,b] \leftrightarrow \int_{[a,-]}(L^1[a,b])$

And it has 10 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$

- derivative Differentiable functions
    - 1 at a point Functions $(a, b) \to \mathbb{R}$ differentiable at a point
    - bd Comparing a function and it derivative
    - cont abs Absolutely continuous functions on $[a, b] \leftrightarrow \int_{[a,-]}(L^1[a, b])$
    - dist Distributional derivatives
    - double circle Double derivative/Laplace operator on the circle
    - frac Fractional derivative
    - limit infinite Infinite limit of derivatives
    - space Space of continuous and continuously differentiable functions on $\mathbb{R}$
    - total Derivative of maps $\mathbb{R}^n \to \mathbb{R}^m$
    - zoom Zooming of a map $\mathbb{R}^n \to \mathbb{R}^m$