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$H^1(U)$

Definition. Sobolev space $H^1(U)$

Let $U(\text{open}) \subseteq \mathbb{R}^n$ be an open set. Then the **Sobolev space** of $L^2(U)$ functions with distributional derivatives in $L^2(U)$

$$H^1(U) := \left\{ f \in L^2(U) \middle| \forall 1 \leq j \leq n, \overset{\int}{\partial}_j f \in L^2(U) \right\}$$

is equipped with the inner product

$$\langle f, g \rangle_{H^1} := \langle f, g \rangle_2 + \sum_{1 \leq j \leq n} \left\langle \overset{\int}{\partial}_j f, \overset{\int}{\partial}_j g \right\rangle_2$$

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is a separable Hilbert space such that

$$\mathrm{Id} \oplus \overset{\int}{\mathrm{grad}} : H^1(U) \to \bigoplus_{j=0}^{n} L^2(U)$$

$$f \mapsto (f, \overset{\int}{\partial}_1 f, \dots, \overset{\int}{\partial}_n f)$$

is a linear isometric embedding.

[1]

Every function in $H^1(a,b)$ has a representative in $\mathcal{C}[a,b]$ and the inclusion

$$H^1(a,b) \hookrightarrow \mathcal{C}[a,b]$$

is continuous.

As

$$f(x) = \int_{[a,x]} f'$$

we have a continuous representative.

- Now

$$|f(x)| \le |f(a)| + \underbrace{\int_{[a,x]} |f'|}_{\le \int_{[a,b]} |f'|}$$

$$\begin{aligned} \|f\|_\infty &\le \inf_{[a,b]} |f| + \int_{[a,b]} |f'| \\ &\le \frac{1}{b-a} \int_{(a,b)} |f| + \int_{[a,b]} |f'| \\ &\le \sqrt{b-a}\|f\|_2 + \sqrt{b-a}\|f'\|_2 \end{aligned}$$

Definition.

$$H_0^1(U) := \overline{\{f \in H^1(U) | \mathrm{supp} f \subset_{\text{cpt}} U\}}^{H^1(U)}$$

Therefore $H_0^1(U)$ is a Hilbert space as well.

Intuition

$$H_0^1(U) = \{f \in H^1(U) \big| f|_{\partial U} \equiv 0\}$$

equivalent inner product

By Poincare inequality

$$\|\mathrm{grad}(u)\|_2^2 \leq \| u \|_{H^1(U)}^2 \leq C\|\mathrm{grad}(u)\|_2^2$$

Let $u \in H^1(U)$ and $g \in L^2(U)$. Then

$$\begin{aligned} &\overset{\int}{\Delta} u = g \\ \iff & -\sum_{j=1}^{n} \left\langle \overset{\int}{\partial}_j -, \overset{\int}{\partial}_j u \right\rangle_{L^2(U)} = \langle -, g\rangle_{L^2(U)} \text{ on } H_0^1(U) \\ \iff & -\langle -, u\rangle_{H^1(U)} \equiv \langle -, g-u\rangle_{L^2} \text{ on } H_0^1(U) \end{aligned}$$

- Let $f \in L^2(U)$. Then we have the functional

$$\langle -, f\rangle_2$$

  on $L^2(U)$.
  - It is continuous on $H_0^1(U)$ because

$$\langle -, f\rangle_2 \leq \|f\|_2 \underbrace{\| - \|_2}_{\leq \| \, \|_{H^1(U)}}$$

- By Riesz representation theorem there exists a $u \in H_0^1(U)$ such that

$$\langle -, f\rangle_2 \equiv \langle -, u\rangle_{H^1} \text{ on } H_0^1(U)$$

  - Therefore,

$$\overset{\int}{\Delta} u = u - f$$

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^2(\Omega)$. Then there exists an unique $H_0^1(U)$ such that

$$\overset{\int}{\Delta} u = u - f$$

- - Let $f \in L^2(U)$. Then we have the functional

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    on $L^2(U)$.

> > - It is continuous on $H_0^1(U)$ because
> >
> > $$\langle -, f\rangle_2 \le \|f\|_2 \underbrace{\|-\|_2}_{\le \|\ \|_{H^1(U)}}$$

- Then by considering the inner product $\langle \mathrm{grad}(-), \mathrm{grad}(-)\rangle_2$, by *Riesz representation* we have an unique $u \in H_0^1(U)$ such that

$$\langle \mathrm{grad}(-), \mathrm{grad}(u)\rangle_2 \equiv \langle -, f\rangle_2 \text{ on } H_0^1(U)$$

- Therefore,

$$-\Delta u \overset{\int}{=} f$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - dist Distributional derivatives
        - H1 $H^1(U)$

And it has 4 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - dist Distributional derivatives
        - H-frac Fractional Sobolev spaces
        - H1 $H^1(U)$
        - H1 loc H^1_{loc}
        - H1 loc R2 $H^1_{\mathrm{loc}}(U)$ for $U \subseteq \mathbb{R}^2$

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1. Lecture04.pdf ↩