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$H^1_{\text{loc}}(U)$ for $U \subseteq \mathbb{R}^2$

Consider

Definition. Local Sobolev space H^1_{loc}

Let $U \subseteq \mathbb{R}^n$ be open. The **local Sobolev space** of functions with derivatives in L^2_{loc}

$$H^1_{\text{loc}}(U) := \left\{ f \in L^2_{\text{loc}}(U) \middle| \forall 1 \leq j \leq n, \overset{\int}{\partial}_j f \in L^2_{\text{loc}}(U) \right\}$$

with norms

$$\|f\|_{H^1(\mathring{K})}$$

for each compact $K \subseteq U$.

for $n = 2$.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ itself we may identify the *functions* with *endomorphisms*

$$H^1_{\text{loc}}(U) \subseteq \text{HomMeas}(U, \mathbb{R}^2)$$

and

$$H^1_{\text{loc}} \cap \mathcal{C}(U) \subseteq \mathcal{C}(U, \mathbb{R}^2)$$

Definition. Let $f \in H^1_{\text{loc}} \cap \mathcal{C}(U)$ then its Jacobian

$$\text{Jac} f := \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2$$

(Change of variables formula) Let $U, V \subseteq \mathbb{C}$ be open and $f : U \to V$ be a proper map in $H^1_{\text{loc}} \cap \mathcal{C}(U)$. Then

$$\forall g \in \mathcal{C}_c(V), (\deg f) \int_V g = \int_U (g \circ f) \text{Jac} f$$

where

Definition. Let $f \in H^1_{\text{loc}} \cap \mathcal{C}(U)$ then its Jacobian

$$\text{Jac} f := \left| \frac{\partial f}{\partial z} \right|^2 - \left| \frac{\partial f}{\partial \bar{z}} \right|^2$$

and

$$\begin{array}{cccc} f^*: & H^2(\text{sing}_c^\bullet(V, \mathbb{Z})) & \to & H^2(\text{sing}_c^\bullet(U, \mathbb{Z})) \\ & \cong & & \cong \\ \deg f: & \mathbb{Z} & \to & \mathbb{Z} \end{array}$$

Let $\text{supp}\, g \subseteq V'(\text{open}) \subseteq V$.

Consider a sequence $\{f_n\} \subseteq \mathcal{C}^\infty(U)$ be such that

$$f_n : f_n^{-1}(V') \to V'$$

is proper of $\deg f$ for all n.

- ... [1](https://rupadarshiray.github.io/notes/wiki.[John_H._Hubbard_-_Teichmüller_Theory_And_Applications_To_Geometry,_Topology,_And_Dynamics._Volume_1_Teichmüller_Theory.pdf#page=141.pdf)]

(Area and the Jacobian) Let $U, V \subseteq \mathbb{C}$ be bounded open sets and let

$$f : U \to V$$

be an orientation-preserving homeomorphism $H^1_{\text{loc}} \cap \mathcal{C}(U)$. Then for any open $W \subseteq U$ we have

$$m(f(U)) = \int_W \text{Jac} f$$

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- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - dist Distributional derivatives
        - H1 loc R2 $H^1_{\text{loc}}(U)$ for $U \subseteq \mathbb{R}^2$

And it has 4 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - derivative Differentiable functions
      - dist Distributional derivatives
        - H-frac Fractional Sobolev spaces
        - H1 $H^1(U)$
        - H1 loc H^1_{loc}
        - H1 loc R2 $H^1_{\text{loc}}(U)$ for $U \subseteq \mathbb{R}^2$

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1. John H. Hubbard - Teichmüller Theory And Applications To Geometry, Topology, And Dynamics. Volume 1 Teichmüller Theory, p.141 ↩