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Quasi-linear PDEs on \mathbb{R}^2

Definition. Definition

Let $O \subseteq \mathbb{R}^3$ be open and $a, b, c \in \mathcal{C}^1(O)$. For $U(\text{open}) \subseteq \pi_{1,2}(O)$ consider

$$Q(a, b, c)(U) := \left\{ u \in \mathcal{C}^1(U) \mid \begin{array}{l} \forall (x, y) \in U \text{ we have } (x, y, u(x, y)) \in O \\ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \end{array} \right\}$$

Bug

Show that this assignment $U \mapsto Q(a, b, c)$ is a sheaf on $\pi_{1,2}(O)$? Donno if that is the case.

- For any $u \in \mathcal{C}^1(U)$, the surface $S(u) := \{z = u(x, y)\}$ has normal $\begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix}$.
- Then

$$\begin{aligned} u \in Q(a, b, c)(U) &\iff \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \\ &\iff (a, b, c)_{(x, y, z)} \in T_{(x, y, z)}S(u) \end{aligned}$$

given an initial curve

- Let Γ_0 be a \mathcal{C}^1 -curve $\gamma : (0, 1) \rightarrow U$ then

$$\begin{aligned} \text{res} : Q(a, b, c)(U) &\rightarrow \mathcal{C}^1(0, 1) \\ u &\mapsto u \circ \gamma \end{aligned}$$

be the restriction.

- Now consider $u_0 \in \mathcal{C}^1(0, 1)$ such that

$$(\gamma, u_0) : (0, 1) \rightarrow O$$

be the *initial curve*.

- We solve for $(X, Y, Z)(s, t)$ in

$$\begin{aligned} \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} &= \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{(X, Y, Z)} \\ (X, Y, Z)_{(s, 0)} &= (\gamma, u_0)(s) \end{aligned}$$

to obtain a \mathcal{C}^1

$$(X, Y, Z) : (0, 1) \times (-\epsilon, \epsilon) \rightarrow O$$

which maps

$$(s, 0) \mapsto (\gamma, u_0)(s)$$

- Let $s_0 \in (0, 1)$ and consider

$$(X, Y) : (0, 1) \times (-\epsilon, \epsilon) \rightarrow \pi_{1,2}(O)$$

whose Jacobian is

$$\begin{bmatrix} X_s & X_t \\ Y_s & Y_t \end{bmatrix}$$

- So, if we assume

$$\det \begin{bmatrix} \gamma'(s_0) & a(\gamma(s_0), u_0(s_0)) \\ & b(\gamma(s_0), u_0(s_0)) \end{bmatrix} \neq 0$$

then there is a onb V of $(X, Y)_{(s_0, 0)}$ inverse

$$(S, T) : V \subseteq \pi_{1,2}(O) \rightarrow (0, 1) \times (-\epsilon, \epsilon)$$

- This means

$$(x, y) \mapsto Z(S(x, y), T(x, y))$$

is in $Q(a, b, c)(V)$.

- This V depends on (γ, u) .

Let $p_0 \in \pi_{1,2}(O)$. Then we can look at all solutions in arbitrary nb of p :

$$Q(a, b, c)_p := \frac{\bigsqcup_{p \in U \subseteq \pi_{1,2}(O)} Q(a, b, c)(U)}{u \sim v \iff \exists V_p : u|_{V_p} = v|_{V_p}}$$

Let $\gamma : (0, 1) \rightarrow \pi_{1,2}(O)$ be a curve such that $\gamma(s_0) = p$. Then we may restrict

$$Q(a, b, c)_{p_0} \rightarrow \mathcal{C}^1(0, 1)_{s_0}$$

$$[u] \mapsto [u \circ \gamma]$$

and reconstruct

$$\left\{ u_0 \in \mathcal{C}^1(0, 1)_{s_0} \mid \det \begin{bmatrix} \gamma'(s_0) & a(\gamma(s_0), u_0(s_0)) \\ & b(\gamma(s_0), u_0(s_0)) \end{bmatrix} \neq 0 \right\} \rightarrow Q(a, b, c)_{p_0}$$

$$u_0 \mapsto u(x, y) := Z(S(x, y), T(x, y))$$

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And it has 3 siblings.

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