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is written (completely with human hands) by Rupadarshi Ray,
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de Rham cohomology in $\mathbb{R}^n$

For any star-shaped subset U of $\mathbb{R}^n$, its de Rham cohomology are

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

By

Star-shaped subsets of $\mathbb{R}^n$ are contractible.

Take a center q of the star-shaped subset, then Id_U is homotopic to the constant map $c_q : x \in U \mapsto q$ by the obvious and natural *straight-line* homotopy:

$$H(x,t) := q + t(x - q)$$

Thus the inclusion $\{q\} \to U$ is a homotopy equivalence.

∎

and

The de Rham cohomology of a contractible set U in a smooth manifold are

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

By

(Homotopy invariance of $H^k(\Omega^\bullet)$) **Given two continuously homotopic, smooth maps**

$$f, g : M \to N$$

between manifolds possibly with boundary, we can always produce a smooth homotopy between them. Two such smoothly homotopic maps induce equal homomorphisms in the cohomology of differential forms

$$f^* = g^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

In particular, if

$$\alpha : M \to N, \beta : N \to M$$

is a smooth homotopy equivalence, then α^*, β^* induce isomorphisms in the cohomology of differential forms.

and

For a point manifold $\{p\}$, the de Rham cohomology

$$H^k_{dR}(\{p\}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

If M is a connected smooth manifold then

$$H^0_{dR}(M) \cong \mathbb{R}$$

- Because there are no $k = -1$-differential forms $\mathrm{im}\,(\mathrm{d}^{k-1}) = 0$.
- $\ker(\mathrm{d}^k) = \{f \in \mathcal{C}^\infty(M) : \mathrm{d}f = 0\}$
  - Since M is connected, $\mathrm{d}f = 0 \iff f = \text{constant} \in \mathbb{R}$
- Thus $H^0_{dR}(M) = \ker(\mathrm{d}^k) = \mathbb{R}$

∎

The rest of the groups vanish because there are no $k > 0$ forms.

∎

∎

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And it has 1 siblings.

- stamp stamp
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