Info

This note found here
as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
created on February 14, 2026 4:19:37 PM,
and was last modified on May 24, 2026 6:41:43 PM.

Harmonic functions composed with conformal maps

Harmonic functions composed with conformal maps is not harmonic in $\dim \geq 3$

Consider the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ with (the metric induced from) the standard metric on $\mathbb{R}^n$. The Laplacian

Definition. We have the Laplace operator

$$\Delta : \mathcal{C}^2 \to \mathcal{C}$$

$$\begin{aligned}\Delta &:= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}(-)\right) + \frac{1}{r^2}\Delta_{S^{n-1}} \\ &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}\end{aligned}$$

Then

$$u(x) = x_1$$

is *harmonic* and the *inversion in S^{n-1}*

$$\begin{aligned} f : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} &\to \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \\ x &\mapsto \frac{x}{|x|^2}\end{aligned}$$

is *conformal*. However

$$(u \circ f)(x) = \frac{x_1}{|x|^2}$$

has Laplacian

$$\begin{aligned}
\Delta(u\circ f) &= \frac{\cancelto{0}{\Delta(x_1)}}{|x|^2} + 2\left\langle \underbrace{\operatorname{grad}(x_1)}_{e_1}, \operatorname{grad}\left(\frac{1}{|x|^2}\right)\right\rangle + x_1\Delta\left(\frac{1}{|x|^2}\right) \\
&= 2\frac{-1}{|x|^4}(2x_1) + x_1\left(\frac{\partial}{\partial r}\frac{-2}{r^3} + \frac{n-1}{r}\left(-\frac{2}{r^3}\right)\right) \\
&= 2\frac{-1}{|x|^4}(2x_1) + x_1\left(\frac{6}{r^4} + \frac{n-1}{r}\left(-\frac{2}{r^3}\right)\right) \\
&= \frac{x_1}{|x|^4}(-4+6-2n+2) \\
&= \frac{(4-2n)x_1}{|x|^4}
\end{aligned}$$

or

$$\begin{aligned}
\Delta(u\circ f) &= \left(\frac{\partial}{\partial x_1}\right)^2\frac{x_1}{|x|^2} && +\sum_{i>1}\left(\frac{\partial}{\partial x_i}\right)^2\frac{x_1}{|x|^2} \\
&= \frac{\partial}{\partial x_1}\frac{1}{|x|^2} - \frac{\partial}{\partial x_1}\frac{2x_1^2}{|x|^4} && -\sum_{i>1}\frac{\partial}{\partial x_i}\frac{x_1}{|x|^4}(2x_i) \\
&= -\frac{2x_1}{|x|^4} - \frac{4x_1}{|x|^4} + \frac{2x_1^2}{|x|^8}(2|x|^2)(2x_1) && -\frac{2x_1}{|x|^4}(n-1) + \frac{x_1}{|x|^8}\sum_{i>1}(4x_i^2)(2|x|^2) \\
&= \frac{x_1}{|x|^4}(-2-4-2n+2) && +8x_1\frac{\left(\sum_{i\geq 1}x_i^2\right)^2}{|x|^8} \\
&= \frac{(4-2n)x_1}{|x|^4}
\end{aligned}$$

So it is not harmonic for $n \geq 3$. [1]

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - harmonic composed conformal Harmonic functions composed with conformal maps

And it has 36 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1

- circle packing Circle packing on $\mathbb{R}^2$
- circle packing to Riemann map Circle packing converges to the Riemann biholomorphism
- Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
- Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
- cont Continuous functions on $\mathbb{R}^d$
- cube dyadic Dyadic cubes
- curves Curves
- derivative Differentiable functions
- forms Differential forms on $\mathbb{R}^n$
- Fourier-Wigner Fourier-Wigner transform
- harmonic composed conformal Harmonic functions composed with conformal maps
- Hilbert Hilbert transform
- hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
- Hol sets Holomorphic subsets of $\mathbb{C}^n$
- hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
- hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
- KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

- Laplace Laplace operator on $\mathbb{R}^n$
- Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- met density Metric density of subsets of $\mathbb{R}^n$
- Mobius n-sphere Mobius endomorphisms
- monotone Monotone functions on $\mathbb{R}$
- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$

- Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$

---

1. https://math.stackexchange.com/a/151838/1290493 ↩