> **Info**
>
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> is written (completely with human hands) by Rupadarshi Ray,
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Laplace's method for approximating integral of e^{Mf} for large M

Let $f : [a, b] \to \mathbb{R}$ have a unique global maximum at $x_0 \in (a, b)$.

> **Example**
>
> $$f(x) = \frac{1}{10}\left(1 - \left(x - \frac{1}{2}\right)^2\right)$$
>
> ```
> f(x)= 0.1(1-(x-0.5)^2)
> e^{Mf(x)}
> M=[1,5,10,20,60]
> ```

> **Naive calculations**
>
> $$f(x) = f(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + O(x-x_0)^3$$
>
> Then
>
> $$\int_{[a,b]} e^{Mf} = e^{Mf(x_0)} \int_{[a,b]} e^{M\frac{(x-x_0)^2}{2}f''(x_0)} \int_{[a,b]} e^{MO(x-x_0)^3}$$
>
> where
>
> $$\int_{[a,b]} e^{M\frac{(x-x_0)^2}{2}f''(x_0)} = \sqrt{\frac{2\pi}{-Mf''(x_0)}}$$
>
> by Gaussian integral.

Let $f \in \mathcal{C}^2[a,b]$ and there exists a unique global maximum $x_0 \in (a,b)$ such that

$$f(x_0) = \max_{x\in[a,b]} f(x),\ f''(x_0) < 0$$

then

$$\lim_{M\to\infty} \frac{\int_{[a,b]} e^{Mf}}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}} = 1$$

- **Lower bound**
  - $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\begin{aligned} x \in B_\delta(x_0) &\implies f''(x_0) - f''(x) \le \epsilon \\ &\implies f''(x) \ge f''(x_0) - \epsilon \end{aligned}$$

  - Because the above inequality is true on $B_\delta(x_0)$ using

(Mean value form of Taylor's theorem) Let $f : [a,b] \to \mathbb{R}$ and a positive integer $n \in \mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there exists $c \in (\alpha, x) \subseteq [a,b]$ such that

$$f(x) = \sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right) + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$

(where the c depends on x).

repeated use of mean value theorem

$$P(x) := \sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right)$$

For a fixed interval $(\alpha, \beta) \subseteq [a,b]$, let M be the number defined by

$$f(x) = P(x) + M(\beta-\alpha)^n$$

and put

$$g(t) := f(t) - P(t) - M(t-\alpha)^n \text{ on } t \in (a,b)$$

Then

$$\begin{aligned} g^{(k)}(t) &= f^{(k)}(t) - P^{(k)}(t) - k!\, M(t-\alpha)^{n-k} \\ g^{(n)}(t) &= f^{(n)}(t) - n!M \text{ on } t \in (a,b) \end{aligned}$$

and for $0 \le k \le n-1$ we have

$$P^{(k)}(\alpha) = f^{(k)}(\alpha) \implies g^{(k)}(\alpha) = 0$$

Also,

$$g(\beta) = 0$$

from definition.

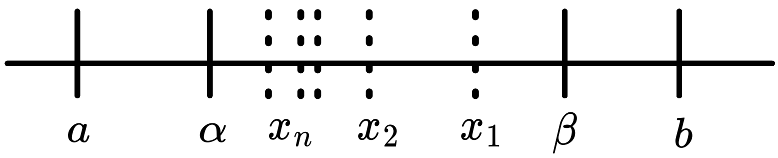
- Hence, as $g(\alpha) = 0 = g(\beta)$, by mean value theorem

$$\exists x_1 \in (\alpha, \beta) : g'(x_1) = 0$$

- Since $g'(\alpha) = 0 = g'(x_1)$, by mean value theorem

$$\exists x_2 \in (\alpha, x_1) : g'(x_2) = 0$$

- After n such steps



we arrive at the conclusion that

$$\exists x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta) : g'(x_n) = 0$$

Hence, there is a $c \in (\alpha, \beta)$ such that

$$f^{(n)}(c) = n!M$$

∎

and same δ gives

$$x \in B_\delta(x_0) \implies f(x) \geq f(x_0) + \frac{(x - x_0)^2}{2}(f''(x_0) - \epsilon)$$

- Because $e^x > 0$ we have

$$\int_{[a,b]} e^{Mf} \geq \int_{B_\delta(x_0)} e^{Mf}$$

which along with

$$\int_{B_\delta(x_0)} e^{Mf(x_0) + \frac{M(x-x_0)^2}{2}(f''(x_0) - \epsilon)}$$
$$= e^{Mf(x_0)} \sqrt{\frac{2\pi}{-M(f''(x_0) - \epsilon)}}$$

gives

$$\epsilon > 0 \implies \frac{\int_{[a,b]} e^{Mf}}{e^{\alpha f^n M f(x_0)} \sqrt{\frac{2\pi}{-Mf''(x_0)}}} \geq \sqrt{\frac{f''(x_0)}{f''(x_0) - \epsilon}}$$

- **Upper bound**
  - Let $f''(x_0) + \epsilon < 0$ then by

(Mean value form of Taylor's theorem) Let $f : [a,b] \to \mathbb{R}$ and a positive integer $n \in \mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there exists $c \in (\alpha, x) \subseteq [a,b]$ such that

$$f(x) = \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k \right) + \frac{f^{(n)}(c)}{n!} (x-\alpha)^n$$

(where the c depends on x).

repeated use of mean value theorem

$$P(x) := \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k \right)$$

For a fixed interval $(\alpha, \beta) \subseteq [a,b]$, let M be the number defined by

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Then

$$g^{(k)}(t) = f^{(k)}(t) - P^{(k)}(t) - k!\, M(t-\alpha)^{n-k}$$
$$g^{(n)}(t) = f^{(n)}(t) - n! M \text{ on } t \in (a,b)$$

and for $0 \leq k \leq n-1$ we have

$$P^{(k)}(\alpha) = f^{(k)}(\alpha) \implies g^{(k)}(\alpha) = 0$$

Also,

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from definition.

- Hence, as $g(\alpha) = 0 = g(\beta)$, by mean value theorem

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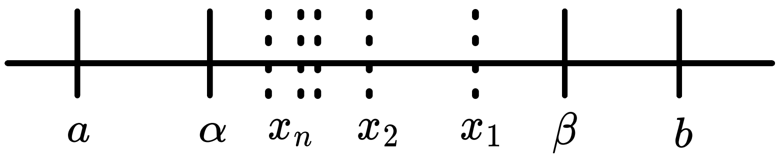
- After n such steps



we arrive at the conclusion that

$$\exists x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta) : g'(x_n) = 0$$

Hence, there is a $c \in (\alpha, \beta)$ such that

$$f^{(n)}(c) = n!M$$

∎

Taylor's theorem, we find $\delta > 0$ such that

$$x \in B_\delta(x_0) \implies f(x) \leq f(x_0) + \frac{(x-x_0)^2}{2}(f''(x_0) + \epsilon)$$

- Because $a, b \in \mathbb{R}$ we have $\eta > 0$ such that

$$x \in [a,b] \setminus B_\delta \implies f(x) \leq f(x_0) - \eta$$

- Now

$$\begin{aligned} \int_{[a,b]} e^{Mf} &\leq \left(\int_{[a,b]\setminus B_\delta} + \int_{B_\delta(x_0)} \right) e^{Mf} \\ &\leq (b-a)e^{M(f(x_0)-\eta)} + \underbrace{\int_{B_{\delta(x_0)}} e^{Mf(x_0) + \frac{M(x-x_0)^2}{2}(f''(x_0)+\epsilon)}}_{\sqrt{\frac{2\pi}{-M(f''(x_0)+\epsilon)}}} \end{aligned}$$

so we have

$$\begin{aligned} \epsilon > 0 \implies & \frac{\int_{[a,b]} e^{Mf}}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}} \\ &\leq (b-a)e^{-M\eta}\sqrt{-M\frac{f''(x_0)}{2\pi}} + \sqrt{\frac{-f''(x_0)}{-f''(x_0)-\epsilon}} \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$

- int
    - Laplace method Laplace's method for approximating integral of e^{Mf} for large M

And it has 4 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - int
            - by parts Integration by parts
            - Laplace method Laplace's method for approximating integral of e^{Mf} for large M
            - open C1bd Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary
            - R Riemann-Darboux and Riemann-Stieltjes integrals