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Riemann-Darboux and Riemann-Stieltjes integrals

Let $[a,b]$ be a compact interval. A **partition** of $[a,b]$ is a finite set of points

$$\{x_k | 1 \le k \le n\} \subset [a,b]$$
$$a = x_0 \le x_1 \le \cdots \le x_n = b$$

in $[a,b]$. The differences are

$$\Delta x_i := x_i - x_{i-1}$$

for each $1 \le i \le n$.

Definition. Darboux integral

Let

$$f : [a,b] \to \mathbb{R}$$

be bounded. Then the **upper Darboux sum** $\overline{\int}_{[a,b]} f$ and **lower Darboux sum** $\underline{\int}_{[a,b]} f$ of f are the following

$$\begin{aligned} &\left(\inf_{[a,b]} f\right)|b-a| \\ &\le \underline{\int}_{[a,b]} f := \sup\left\{ \sum_{1\le k\le n} \left(\inf_{[x_{i-1},x_i]} f\right)\Delta x_i \,\middle|\, \begin{array}{r} \{x_0,\dots,x_n\} \subset [a,b] \\ \text{is a partition} \\ \text{for some } n \ge 1 \end{array} \right\} \\ &\le \overline{\int}_{[a,b]} f := \inf\left\{ \sum_{1\le k\le n} \left(\sup_{[x_{i-1},x_i]} f\right)\Delta x_i \,\middle|\, \begin{array}{r} \{x_0,\dots,x_n\} \subset [a,b] \\ \text{is a partition} \\ \text{for some } n \ge 1 \end{array} \right\} \\ &\le \left(\sup_{[a,b]} f\right)|b-a| \end{aligned}$$

If the upper and lower Darboux sum are equal, then the **Darboux integral** of f on $[a,b]$ is

$$\int_{[a,b]} f := \overline{\int}_{[a,b]} f = \underline{\int}_{[a,b]} f$$

Definition. Riemann integral

integration and differentiation

Let f be Riemann-Darboux integrable on $[a,b]$ and let F be differentiable on $[a,b]$ such that $F' \equiv f$. Then

$$\int_{[a,b]} f = F(b) - F(a)$$

for Riemann integral

for Darboux integral

Let $\epsilon > 0$. Then as f is Riemann-Darboux integrable there exists a parition $\{x_0, \dots, x_n\} \subset [a,b]$ so that

$$\left| \sum_{1 \le k \le n} \left(\inf_{[x_{i-1},x_i]} f \right) \Delta x_i - \sum_{1 \le k \le n} \left(\inf_{[x_{i-1},x_i]} f \right) \Delta x_i \right| < \epsilon$$

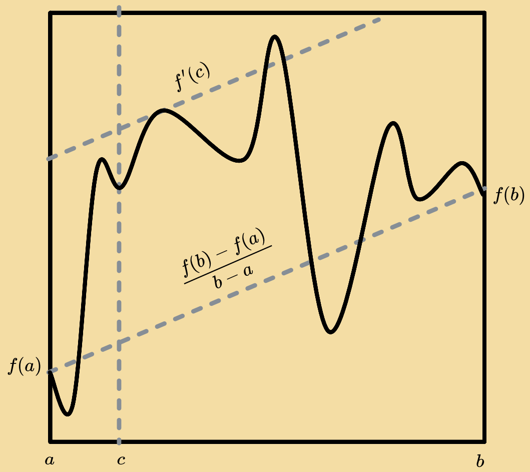
- Then by

(Lagrange's mean value theorem) **Let a continuous function**

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b)**. Then there is a point** $c \in (a, b)$ **at which**

$$f(b) - f(a) = (b - a)f'(c)$$



there are points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

for $1 \leq i \leq n$.

- Thus

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$$

- By

6.7 Theorem

(*a*) *If* (13) *holds for some* P *and some* ε, *then* (13) *holds* (*with the same* ε) *for every refinement of* P.

(*b*) *If* (13) *holds for* $P = \{x_0, \ldots, x_n\}$ *and if* s_i, t_i *are arbitrary points in* $[x_{i-1}, x_i]$, *then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i < \varepsilon.$$

(*c*) *If* $f \in \mathscr{R}(\alpha)$ *and the hypotheses of* (*b*) *hold, then*

$$\left| \sum_{i=1}^{n} f(t_i) \, \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon.$$

Proof Theorem 6.4 implies (*a*). Under the assumptions made in (*b*), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \leq M_i - m_i$. Thus

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (*b*). The obvious inequalities

$$L(P, f, \alpha) \leq \sum f(t_i) \, \Delta\alpha_i \leq U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \leq \int f \, d\alpha \leq U(P, f, \alpha)$$

prove (*c*).

we have

$$\left| F(b) - F(a) - \int_{[a,b]} f \right| < \epsilon$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - int
      - R Riemann-Darboux and Riemann-Stieltjes integrals

And it has 4 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - int
      - by parts Integration by parts
      - Laplace method Laplace's method for approximating integral of e^{Mf} for large M
      - open C1bd Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary

- R Riemann-Darboux and Riemann-Stieltjes integrals