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is written (completely with human hands) by Rupadarshi Ray,
created on June 10, 2026 6:17:27 PM,
and was last modified on June 12, 2026 7:51:33 PM.

Integration by parts

- Let $f, g \in \mathcal{C}^1[a,b]$ then

$$(fg)' = f'g + g'f$$

and

Let f be Riemann-Darboux integrable on $[a,b]$ and let F be differentiable on $[a,b]$ such that $F' \equiv f$. Then

$$\int_{[a,b]} f = F(b) - F(a)$$

implies

$$f(b)g(b) - f(a)g(a) = \int_{[a,b]} (f'g + fg')$$

We may assume f, g to be absolutely continuous.

(Fundamental theorem of analysis, in the almost everywhere sense) A function

$$F : [a,b] \to \mathbb{R}$$

is *absolutely continuous* on $[a,b] \implies F'$ exists m-almost everywhere and is L^1. In that case,

$$F(x) = F(a) + \int_{[a,x]} F' \qquad \text{on } [a,b]$$

Moreover $F' = 0$ m-almost everywhere $\implies F$ is constant m-almost everywhere.
Conversely, for every $f \in L^1[a,b]$ there exists a function F which is differentiable m-almost everywhere and

$$F' = f \text{ ae}$$

and in fact we may take F to be the absolutely continuous function $x \mapsto c_0 + \int_{[a,x]} f$ for any $c_0 \in \mathbb{C}$.

for absolutely continuous functions

However, we do not need f, g to be *differentiable* for the integration by parts to be true.

- Let

$$f, g : [a, b] \to \mathbb{C}$$

be such that there exists $\phi, \psi \in L^1[a, b]$ satisfying

$$f(x) = c_1 + \int_{[a,x]} \phi$$
$$g(x) = c_2 + \int_{[a,x]} \psi$$

for $c_1, c_2 \in \mathbb{R}$. In particular, $f, g \in \mathcal{C}[a, b]$.

- As $\phi, \psi \in L^1[a, b]$ we have

$$(x, y) \mapsto \phi(x)\psi(y)$$

in $L^1[a, b]^2$.

- By

(Fubini's theorem) Suppose $f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then for almost every $y \in \mathbb{R}^{d_2}$

$$f(-, y) \in L^1(\mathbb{R}^{d_1})$$

and

$$y \mapsto \int_{\mathbb{R}^{d_1}} f(-, y) \in L(\mathbb{R}^{d_2})$$

Moreover,

$$\int_{y \in \mathbb{R}^{d_2}} \left(\int_{x \in \mathbb{R}^{d_1}} f(x, y) \right) = \int_{\mathbb{R}^{d_1 + d_2}} f$$

$$
\begin{aligned}
\int_{[a,b]}(f\phi+g\psi) &= \int_{t\in[a,b]} g(t)\int_{s\in[a,t]} f(s) + \int_{t\in[a,b]} f(t)\int_{s\in[a,t]} g(s) \\
&= \int_{[a,b]\times[a,b]} f\otimes g \\
&= \left(\int_{[a,b]} f\right)\left(\int_{[a,b]} g\right) \\
&= (f(b)-f(a))(g(b)-g(a))
\end{aligned}
$$

when $f'g$ and fg' are not L^1

[1]

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - int
            - by parts Integration by parts

And it has 4 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - int
            - by parts Integration by parts
            - Laplace method Laplace's method for approximating integral of e^{Mf} for large M
            - open C1bd Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary
            - R Riemann-Darboux and Riemann-Stieltjes integrals

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1. https://faculty.uml.edu/jpropp/parts.pdf ↩