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>
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> is written (completely with human hands) by Rupadarshi Ray,
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Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary

object	In Cartesian coordinates
Volume form $\lambda_{\mathbb{R}^n} \equiv \lambda$	$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$

integrating *gradient vector field* $\otimes$ *volume form* in $\mathbb{R}^n$

> **Let $B \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Then**
>
> $$\int_B \operatorname{grad}(u) \otimes \lambda_B = \int_{\partial B} u\, \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

using Stroke's theorem

- Let $B \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Let $u \in \mathcal{C}^1(\overline{B})$ and consider

  $$^{n-1}\alpha = u\widehat{\mathrm{d}x_i}$$

  where $\widehat{\mathrm{d}x_i}$ is $\lambda = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $\mathrm{d}x_i$ removed.
  - The **exterior derivative** of this $n-1$ form is the n-form

    $$\mathrm{d}\alpha = \frac{\partial u}{\partial x_i}\mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} = (-1)^{i+1}\frac{\partial u}{\partial x_i}\lambda$$

    where

    $$\begin{aligned}\mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} &= \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \ldots \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n \\ &= (-1)^{i+1}\underbrace{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}_{\lambda}\end{aligned}$$

- Then by *Stroke's theorem*

$$\begin{aligned}\int_B \mathrm{d}\alpha &= (-1)^{i+1}\int_{\partial B}\alpha \\ \implies \int_B \frac{\partial u}{\partial x_i}\lambda &= (-1)^{i+1}\int_{\partial B} u\widehat{\mathrm{d}x_i}\end{aligned}$$

Example

For $n = 2$ we have

$$\int_{\partial B} -u\mathrm{d}x = \int_B \frac{\partial u}{\partial y}\mathrm{d}x \wedge \mathrm{d}y$$

- Let us parameterize ∂B by a $\mathcal{C}^1$ function

$$\begin{aligned}U(\text{open}) \subseteq \mathbb{R}^{n-1} &\to B \\ y &\mapsto (g(y), y)\end{aligned}$$

  where $g \in \mathcal{C}^1(U, \mathbb{R})$.

  - Then

$$\begin{aligned}\int_{\partial B} u\widehat{\mathrm{d}x_i} &= \int_U u(g(y), y)\mathrm{d}g \wedge \widehat{\mathrm{d}y_{i-1}} \\ &= \int_U u(g(y), y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_{i-1} \wedge \widehat{\mathrm{d}y_{i-1}} \\ &= (-1)^i \int_U u(g(y), y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}\end{aligned}$$

    for $i > 1$ and

$$= \int_U u(g(y), y)\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}$$

    for $i = 1$.

  - Thus

$$\int_B \frac{\partial u}{\partial x_i}\lambda = \begin{cases} \int_U u(g(y), y)\mathrm{d}y_1 \wedge \ldots . \wedge \mathrm{d}y_{n-1} & i = 1 \\ -\int_U u(g(y), y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1} & i < n \end{cases}$$

  - Tensoring with $\hat{e}_i$ and summing for $1 \leq i \leq n$ we have

$$\int_B \mathrm{grad}(u) \otimes \lambda = \int_U u(1, -\mathrm{grad}(g)) \otimes (\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1})$$

    where $X \otimes \lambda$ is a vector field $\otimes$ n-form, which we understand is integrated component-wise.

- We identify

$$(1, -\mathrm{grad}(g)) \otimes (\mathrm{d}y_1 \wedge \ldots . \wedge \mathrm{d}y_{n-1}) =: \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

  then we have

$$\int_B \mathrm{grad}(u) \otimes \lambda_B = \int_{\partial B} u\, \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

integration by parts in $\mathbb{R}^n$

(Integration by parts in $\mathbb{R}^n$) Let $u, v \in \mathcal{C}^1(\overline{U})$. Then

$$\int_U u_{x_i} v = -\int_U u v_{x_i} + \int_{\partial U} u v \hat{n}_i$$

integral of divergence

(Integral of divergence) Let $u \in \mathcal{C}^2(\overline{U})$. Then ›

$$\int_U \Delta u = \int_{\partial U} \frac{\partial u}{\partial \hat{n}}$$

Intuition

The Laplacian Δu measures how much $\mathrm{grad}(u)$ moves points outside of U, however it must "move it" through the boundary by amount $\mathrm{grad}(u) \cdot \hat{n}$ therefore

(Integral of divergence) Let $u \in \mathcal{C}^2(\overline{U})$. Then ›

$$\int_U \Delta u = \int_{\partial U} \frac{\partial u}{\partial \hat{n}}$$

Using

(Integration by parts in $\mathbb{R}^n$) Let $u, v \in \mathcal{C}^1(\overline{U})$. Then

$$\int_U u_{x_i} v = -\int_U u v_{x_i} + \int_{\partial U} u v \hat{n}_i$$

we have

$$\int_U u_{x_i x_i} = \int_{\partial U} u_{x_i} \hat{n}_i$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - int
      - open C1bd Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary

And it has 4 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - int
      - by parts Integration by parts
      - Laplace method Laplace's method for approximating integral of e^{Mf} for large M
      - open C1bd Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary
      - R Riemann-Darboux and Riemann-Stieltjes integrals