

## Info

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is written (completely with human hands) by [Rupadarshi Ray](#),  
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# Mapping balls

## Definition. Derivative of a map $U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ at a point

Let

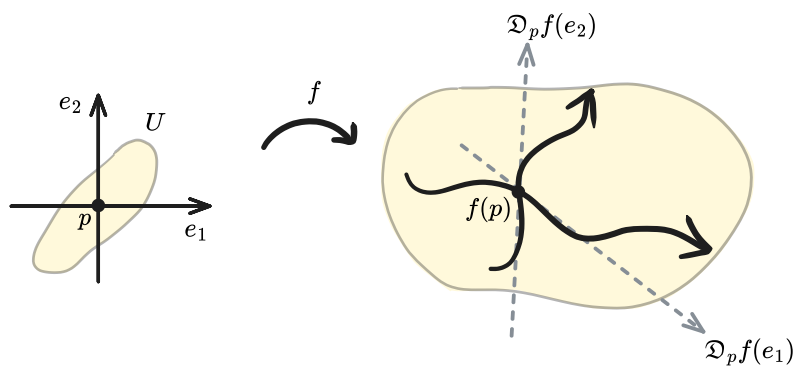
$$f : U \text{ (open)} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

be a map. Then the **derivative** of  $f$  at  $p \in U$  is the linear map

$$\mathcal{D}_p f : T_p \mathbb{R}^n \rightarrow T_p \mathbb{R}^m$$

such that

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{f(p+v) - f(p) - \mathcal{D}_p f(v)}{|v|} &= 0 \\ \iff \lim_{v \rightarrow 0} \frac{|f(p+v) - f(p) - \mathcal{D}_p f(v)|}{|v|} &= 0 \end{aligned}$$



$$\iff \lim_{x \rightarrow p} \frac{|f(x) - f(p) - \mathcal{D}_p f(x-p)|}{|x-p|} = 0$$

$$\iff \forall \epsilon > 0 \exists r(\epsilon) > 0 :$$

$$\forall x \in B_{r(\epsilon)}(p), |f(x) - f(p) - \mathcal{D}_p f(x-p)| \leq \epsilon |x-p|$$

- This implies for  $x \in B_{r(\epsilon)}(p)$  we have

$$\begin{aligned}
|f(x) - f(p)| &= |f(x) - f(p) - \mathfrak{D}_p f(x-p) + \mathfrak{D}_p f(x-p)| \\
&\leq |f(x) - f(p) - \mathfrak{D}_p f(x-p)| + |\mathfrak{D}_p f(x-p)| \\
&\leq \epsilon |x-p| + \|\mathfrak{D}_p f\| |x-p| \\
&\leq (\epsilon + \|\mathfrak{D}_p f\|) r(\epsilon)
\end{aligned}$$

- Therefore,  $f$  takes the ball of radius  $r(\epsilon)$  about  $p$  into the ball of radius  $(\epsilon + \|\mathfrak{D}_p f\|)r(\epsilon)$  about  $f(p)$

$$B_{r(\epsilon)}(p) \xrightarrow{f} B_{(\epsilon + \|\mathfrak{D}_p f\|)r(\epsilon)}(f(p))$$

### for $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ with invertible $\mathfrak{D}_p f$

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- For all  $x \in B_{r(\epsilon)}(p)$  we have

$$\begin{aligned}
|f(x) - f(p)| &\geq |\mathfrak{D}_p f(x-p)| - |f(x) - f(p) - \mathfrak{D}_p f(x-p)| \\
&\geq \frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} |x-p| - \epsilon |x-p| \\
&\geq \left( \frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} - \epsilon \right) |x-p|
\end{aligned}$$

- Let  $\epsilon > 0$  be small enough so that

$$a(\epsilon) := \frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} - \epsilon > 0$$

- For  $x \in \partial B_{r(\epsilon)}(p)$ , we have

$$|f(x) - f(p)| \geq \underbrace{a(\epsilon)r(\epsilon)}_{>0}$$

so

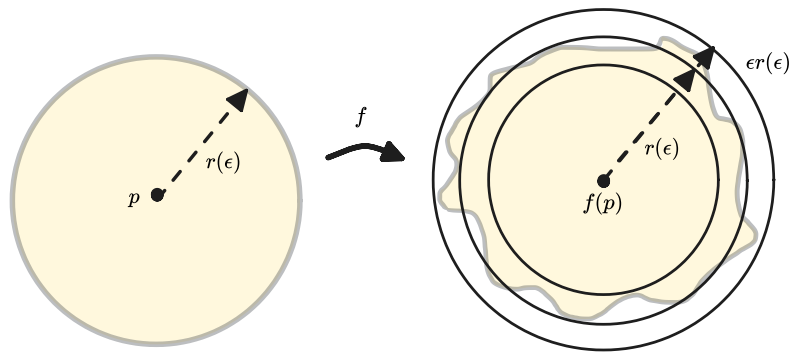
$$\overline{\partial B_{r(\epsilon)}(p)} \xrightarrow{f} (B_{a(\epsilon)r(\epsilon)}(f(p)))^c$$

### for $\mathfrak{D}_p f = \text{Id}$

- For the case  $\mathfrak{D}_p f = \text{Id}$ , consider  $\epsilon \in (0, 1)$ , then we have

$$\begin{aligned}
B_{r(\epsilon)}(p) &\xrightarrow{f} B_{(\epsilon+1)r(\epsilon)}(f(p)) \\
\partial B_{r(\epsilon)}(p) &\xrightarrow{f} (B_{(1-\epsilon)r(\epsilon)}(f(p)))^c
\end{aligned}$$

- Therefore,  $f(x)$  is strictly different from  $f(p)$  outside the ball of radius  $(1 - \epsilon)r(\epsilon)$  about  $f(p)$ .



### Intuition

The ball  $B_{(1-\epsilon)r(\epsilon)}(f(p))$  seems to be inside the image  $f(B_{r(\epsilon)}(p))$  by the [above picture](#).

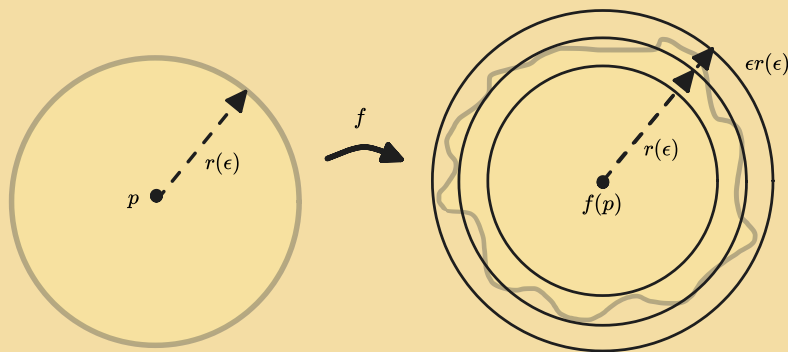
And it is.

Let  $p \in U$  and  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a map such that  $f(p) = p$ ,  $\mathcal{D}_p f = \text{Id}$ , that is,

$$\forall \epsilon \exists r(\epsilon) : \forall x \in B_{r(\epsilon)}(p), |f(x) - f(p) - x + p| \leq \epsilon |x - p|$$

Then for each  $r \in (0, r(\epsilon)]$

$$B_{(1-\epsilon)r}(f(p)) \subseteq f(B_r(p))$$



For  $r \in (0, r(\epsilon)]$  and  $x \in B_r(p)$  and  $y \in B_{(1-\epsilon)r}(p)$  we have

$$\begin{aligned} |x - f(x) + y - p| &\leq |x - f(x)| + |y - p| \\ &\leq \epsilon |x - p| + |y - p| \\ &\leq \epsilon r + (1 - \epsilon)r \\ &= r \end{aligned}$$

• Then

$$\begin{aligned} \mathfrak{F}_y f : \overline{B_r(p)} &\rightarrow B_r(p) \\ x &\mapsto x - f(x) + y \end{aligned}$$

• By

**(Brouwer's fixed point theorem)** Let  $\varphi : \overline{D^n} \rightarrow \overline{D^n}$  be a continuous map for  $n \geq 2$ , then it has a fixed point.

we conclude  $\mathfrak{B}_y f$  has a fixed point in  $\overline{B_r(p)}$ .

- Then

$$\exists x : x = x - f(x) + y \implies f(x) = y$$

- Thus,

$$B_{(1-\epsilon)r}(p) \subseteq f(B_r(p))$$

for conformal  $\mathcal{D}_p f$

for  $\mathbb{C}$ -linear  $\mathcal{D}_p f$

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