> **Info**
>
> This note found here
> as a part of a collection
> is written (completely with human hands) by Rupadarshi Ray,
> created on November 30, 2024 9:36:10 PM,
> and was last modified on June 12, 2026 11:24:16 AM.

Inverse of differentiable maps

differentiable at a point

> **Example**
>
> Even if a function is smooth everywhere except a differentiable point, it may not be locally injective.
>
> > **Proposition:** The differentiable map
> >
> > $$\begin{aligned}\mathbb{R} &\to \mathbb{R}\\ x &\mapsto x + x^2 \sin\frac{1}{x}\end{aligned}$$
> >
> > ```
> > left = -1
> > right = 1
> > top = 1
> > bottom = -1
> > ---
> > x+x^2 \sin(1/x)
> > ```
> >
> > is $\mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ and has derivative 1 at 0, however it not injective on any onb of 0.

for $\mathcal{C}^1$ maps

> **(Inverse function theorem)** Let
>
> $$f : E \text{ (open)} \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

be $\mathcal{C}^1(E)$ with $\mathfrak{D}_a f$ invertible for some $a \in E$. Then there is a open subset $U \subseteq E$ such that f is one-one on U, $f(U)$ is open and

$$f^{-1} \in \mathcal{C}^1(f(U))$$

fixed points of the pre-shifters

- As

$$\mathfrak{D} f : E \to \mathsf{Mat}_{\mathbb{R}}(n)$$

is a **continuous** map there is an open ball $B(a) \subseteq E$ such that $\forall x \in B(a)$ we have

$$\|\mathfrak{D}_x f - \mathfrak{D}_a f\| < \frac{1}{2\,\|(\mathfrak{D}_a f)^{-1}\|}$$

Definition. Pre-shifter of a map

Let

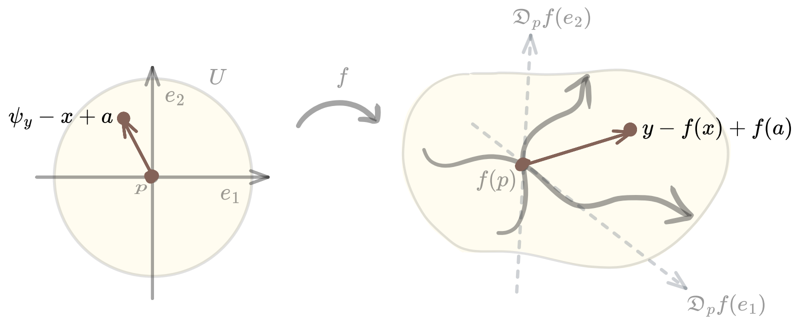
$$f : E\,(\text{open}) \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

be such that $\mathfrak{D}_a f$ is invertible for some $a \in E$. For each $y \in \mathbb{R}^n$, the **pre-shifter** at y is the map

$$\begin{aligned} \mathfrak{P}_y f &: E \to \mathbb{R}^n \\ \mathfrak{P}_y f(x) &:= x + (\mathfrak{D}_a f)^{-1}(y - f(x)) \\ &= x - a + \mathfrak{Z}_a^{-1}(y - f(x) + f(a)) \end{aligned}$$

that is, *pre-shifter* (at y) of x is the unique vector $\mathfrak{P}_y f(x)$ such that

$$\begin{aligned} &\mathfrak{P}_y f(x) - x \xmapsto{\mathfrak{D}_a f} y - f(x) \\ \iff &\mathfrak{P}_y f(x) - x + a \xmapsto{\mathfrak{Z}_a f} y - f(x) + f(a) \end{aligned}$$



Intuition

The pre-shifter of f (at y) of x is the *approximate* point x must be shifted to, so that f *approximately* maps x to y.

Proposition: A map f sends x to y $\iff$ the pre-shifter of f (at y) has x as a fixed point

$$f(x) = y \iff \mathfrak{P}_y f(x) = x$$

- -
$$(\mathfrak{D}_a f)(\mathfrak{P}_y f(x) - x) = y - f(x)$$

  - Since

$$\begin{aligned}\mathfrak{D}_x \mathfrak{P}_y f &= \mathsf{Id} - (\mathfrak{D}_a f)^{-1} \mathfrak{D}_x f \\ &= (\mathfrak{D}_a f)^{-1}(\mathfrak{D}_a f - \mathfrak{D}_x f)\end{aligned}$$

    so by

    - As

$$\mathfrak{D} f : E \to \mathsf{Mat}_{\mathbb{R}}(n)$$

      is a **continuous** map there is an open ball $B(a) \subseteq E$ such that $\forall x \in B(a)$ we have

$$\|\mathfrak{D}_x f - \mathfrak{D}_a f\| < \frac{1}{2\,\|(\mathfrak{D}_a f)^{-1}\|}$$

$$x \in B(a) \implies \|\mathfrak{D}_x \mathfrak{P}_y f\| < \frac{1}{2}$$

  - Hence, by

    Let

$$f : U \text{ (open, convex)} \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

    be differentiable

$$\mathfrak{D} f : U \to \mathrm{EndVec}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$$

    such that the operator norm of the derivative on U is bounded by M

$$\forall x \in U, \|\mathfrak{D}_x f\|_{n \to m} \leq M$$

Then f is M-Lipshitz on U

$$\forall x, y \in U, |f(y) - f(x)| \leq M |y - x|$$

we have

$$\forall x_1, x_2 \in B(a), \|\mathfrak{P}_y f(x_1) - \mathfrak{P}_y f(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$$

- Thus, $\mathfrak{P}_y f$ has at most one fixed point in $B(a)$ so $f|_{B(a)}$ is **one-one**.
- Choose $f(x_0) = y_0 \in f(B(a))$ such that and a closed ball $\overline{B_\beta(x_0)} \subset B(a)$.
  - Fix $y \in \mathbb{R}^n$ such that

    $$\|y - y_0\| < \frac{r}{2\|\mathfrak{D}_a f\|}$$

    then

    $$\|\mathfrak{P}_y f(x_0) - x_0\| = \left\|(\mathfrak{D}_a f)^{-1}(y - y_0)\right\| < \frac{r}{2}$$
    $$x \in \overline{B_\beta(x_0)} \implies \|\mathfrak{P}_y f(x) - x_0\| \leq \|\mathfrak{P}_y f(x) - \mathfrak{P}_y f(x_0)\| + \underbrace{\|\mathfrak{P}_y f(x_0) - x_0\|}_{<\frac{r}{2}}$$
    $$\leq \frac{1}{2}\|x - x_0\| + \frac{r}{2}$$
    $$\leq r$$

    Hence, $\mathfrak{P}_y f(x) \in B_\beta(x_0)$.
  - Thus,

    $$\mathfrak{P}_y f : \overline{B_\beta(x_0)} \to \overline{B_\beta(x_0)}$$

    is a contraction on a complete metric space, so there exists a fixed point x of $\mathfrak{P}_y f$ $\implies f(x) = y$
  - So each $y_0 \in f(B(a))$ has a ball around it inside $f(B(a))$, so $f(B(a))$ **is open**.

for maps differentiable everywhere

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - mapping

            - inverse Inverse of differentiable maps

And it has 4 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - mapping
            - balls Mapping balls
            - inverse Inverse of differentiable maps
            - mean value Mean value property of functions
            - seqf Sequence of functions on $\mathbb{R}^d$