

 Info

This note [found here](#)
as a part of [a collection](#)
is written (completely with human hands) by [Rupadarshi Ray](#),
created on July 11, 2024 1:45:01 AM,
and was last modified on June 12, 2026 11:24:16 AM.

n -times differentiable functions

functions n -times differentiable at a point

We generalize the preliminary result

• From

• Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable at $c \in (a, b)$.

• Define

$$f_c^*(x) := \begin{cases} \frac{f(x) - f(c)}{x - c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

• f^* is continuous in (a, b) .

• So we can write f as

$$f(x) - f(c) = (x - c)f_c^*(x)$$

for an unique continuous f_c^* .

we write

$$f(x) = f(c) + (x - c)f_c^*(x)$$

as

$$f(x) = f(c) + (x - c)f'(c) + (x - c)h_c(x)$$

where

$$h_c(x) := f_c^*(x) - f'(c) \\ \implies \lim_{x \rightarrow c} h_c(x) = 0$$

for functions that are n -times differentiable at a point

☰ **(Peano form of Taylor's theorem)** Let $f : (a, b) \rightarrow \mathbb{R}$ and a positive integer $n \in \mathbb{Z}_{>0}$ such that f is n -times differentiable at $c \in (a, b)$. Then there exists a function $h_c : (a, b) \rightarrow \mathbb{R}$ such that

$$f(x) = \sum_{k=0}^n \left(\frac{(x-c)^k}{k!} f^{(k)}(c) \right) + (x-c)^n h_c(x)$$

and

$$\lim_{x \rightarrow c} h_c(x) = 0$$

functions n -times differentiable on an interval

By using the *mean value theorem*

Corollary of mean value theorem: If

$$f : (a, b) \rightarrow \mathbb{R}$$

is differentiable, then for any proper subinterval $(x_1, x_2) \subset (a, b)$ by mean value theorem we have

$$\exists x \in (x_1, x_2) : f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$$

repeatedly, we obtain

☰ **(Mean value form of Taylor's theorem)** Let $f : [a, b] \rightarrow \mathbb{R}$ and a positive integer $n \in \mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}$ exists on (a, b) . Then there exists $c \in (\alpha, x) \subseteq [a, b]$ such that

$$f(x) = \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k \right) + \frac{f^{(n)}(c)}{n!} (x - \alpha)^n$$

(where the c depends on x).

repeated use of mean value theorem

$$P(x) := \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k \right)$$

For a fixed interval $(\alpha, \beta) \subseteq [a, b]$, let M be the number defined by

$$f(x) = P(x) + M(\beta - \alpha)^n$$

and put

$$g(t) := f(t) - P(t) - M(t - \alpha)^n \text{ on } t \in (a, b)$$

Then

$$\begin{aligned} g^{(k)}(t) &= f^{(k)}(t) - P^{(k)}(t) - k! M(t - \alpha)^{n-k} \\ g^{(n)}(t) &= f^{(n)}(t) - n!M \text{ on } t \in (a, b) \end{aligned}$$

and for $0 \leq k \leq n - 1$ we have

$$P^{(k)}(\alpha) = f^{(k)}(\alpha) \implies g^{(k)}(\alpha) = 0$$

Also,

$$g(\beta) = 0$$

from definition.

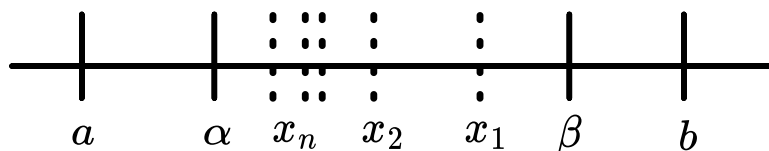
- Hence, as $g(\alpha) = 0 = g(\beta)$, by mean value theorem

$$\exists x_1 \in (\alpha, \beta) : g'(x_1) = 0$$

- Since $g'(\alpha) = 0 = g'(x_1)$, by mean value theorem

$$\exists x_2 \in (\alpha, x_1) : g'(x_2) = 0$$

- After n such steps



we arrive at the conclusion that

$$\exists x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta) : g'(x_n) = 0$$

Hence, there is a $c \in (\alpha, \beta)$ such that

$$f^{(n)}(c) = n!M$$

if first $n - 1$ derivatives are zero at a point

If

$$f^{(k)}(\alpha) = 0$$

then

$$\begin{aligned}\exists c \in (\alpha, x) : f(x) &= \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k \right) + \frac{f^{(n)}(c)}{n!} (x - \alpha)^n \\ &= \frac{f^{(n)}(c)}{n!} (x - \alpha)^n\end{aligned}$$

if n -th derivative is continuous and non-zero at a point

- Let

$$\begin{aligned}f &\in \mathcal{C}^n[a, b] \\ \exists x_0 &\in [a, b] : f^{(n)}(x_0) < 0\end{aligned}$$

- Then by continuity of $f^{(n)}$, for any $\epsilon > 0$ we have a $\delta > 0$ such that

$$|f^{(n)}(x) - f^{(n)}(x_0)| \leq \epsilon$$

S

Current note has 0 direct children and 0 total descendants.

- [stamp](#) stamp
 - [Rf](#) subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
 - [mapping](#)
 - [mean value](#) Mean value property of functions
 - [iterative](#) Taylor's theorem

And it has 1 siblings.

- [stamp](#) stamp
 - [Rf](#) subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
 - [mapping](#)
 - [mean value](#) Mean value property of functions
 - [iterative](#) Taylor's theorem