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Mean value property of functions

From

(Extremum value theorem for continuous $[a,b] \to \mathbb{R}$) Let

$$f : [a,b] \to \mathbb{R}$$

be continuous, then a global maxima *and* a minima exists $M, m \in [a,b]$ for f.

By

A continuous function from a compact metric space to $\mathbb{R}$ has a global maxima *and* a global minima.

- Image is a compact subset of $\mathbb{R}$, so it closed and bounded, so its sup and inf is in the set.
- The sup and inf are global maxima and minima.

we already know any continuous function on $[a,b]$ has a global maxima and a global minima.

There are two possibilities

- at least **one extremum is in the interior** (a,b)
- maxima and the minima **both extremums are at the boundary points** a, b

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    left=-1; right=2;
    top=1; bottom=-1;
---
0.5x
0.5(1-(-0.5+x)^2)
```

Obviously it is possible to have $f' \neq 0$ on $[a,b]$ for a continuous function on $[a,b]$ that is differentiable function on (a,b), just take $f(x) = x$ on $[0,1]$.

$\mathcal{C}[a,b]$ functions with fixed endpoints differentiable on (a,b) have at least one stationary point

But what if we demand the endpoints have the same value. This defines a function

$$\begin{array}{r} f:[a,b]\to\mathbb{R} \\ f(a)=f(b) \end{array} \leftrightarrow f: S^1 \to \mathbb{R}$$

on the circle

$$\frac{[a,b]}{a\sim b} \cong_{\text{Man}} S^1$$

but the hypothesis of f being differentiable on (a,b), this just says f is differentiable on S^1 except at one point.

(Rolle's theorem extended by Cauchy) Let a continuous

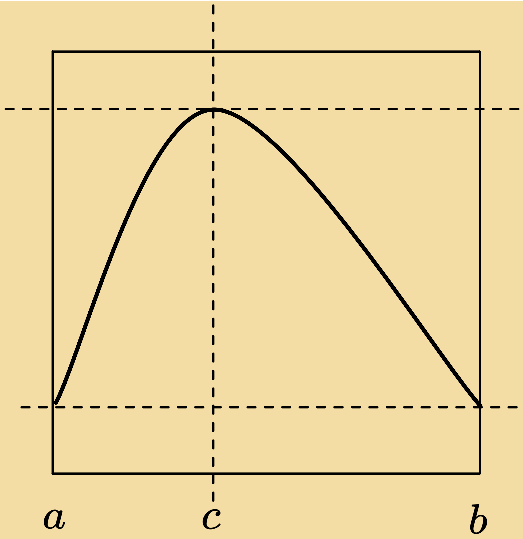
$$\begin{array}{r} f:[a,b]\to\mathbb{R} \\ f(a)=f(b) \end{array}$$

such that

$$f' : (a,b) \to \mathbb{R} \cup \{\infty, -\infty\}$$

exists. Then there is a point in the interior where the derivative is zero

$$\exists c \in (a,b) \text{ such that } f'(c) = 0$$



- **(Extremum value theorem for continuous $[a,b] \to \mathbb{R}$) Let**

  $$f : [a,b] \to \mathbb{R}$$

  be continuous, then a global maxima *and* a minima exists $M, m \in [a,b]$ for f.

  By

  A continuous function from a compact metric space to $\mathbb{R}$ has a global maxima *and* a global minima.

  - Image is a compact subset of $\mathbb{R}$, so it closed and bounded, so its sup and inf is in the set.
  - The sup and inf are global maxima and minima.

- Define M, m such that $f(M) = \sup f$ and $f(m) = \inf f$
  - If one of M, m is in the interior (a,b) then $f'(m) = 0$ or $f'(M) = 0$ at that extremum value because of

    (Derivative, if it exists, must be 0 at local extremas) Let a function $f : (a,b) \to \mathbb{R}$ have a local maxima or minima at $c \in (a,b)$. If f has an extended derivative at c then the derivative must be zero

    $$f'(c) \in \mathbb{R} \cup \{-\infty, \infty\} \implies f'(c) = 0$$

Therefore,

$$\{\text{local extremum of } f\} \subseteq Z(f')$$

By

(Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ and

$$f'(c) > 0 \text{ or } +\infty$$

then there is a $I(c) \subseteq (a, b)$ in which $f(x) - f(c)$ has same sign as $x - c$, that is, f is strictly increasing on $I(c)$.

- If $f'(c) > 0$, then by
  - Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.
    - Define

      $$f_c^*(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

      - f^* is continuous in (a, b).
    - So we can write f as

      $$f(x) - f(c) = (x - c) f_c^*(x)$$

      for an unique continuous f_c^*.

  we have

  $$f(x) - f(c) = (x - c) f_c^*(x)$$

  for a continuous f_c^*
- By

  (Continuity preserves sign locally): Let $I(c)$ be an interval containing c and

  $$f : I(c) \subset \mathbb{R} \to \mathbb{R}$$

  be continuous at $c \in I(c), f(c) \neq 0$. Then for $\epsilon := |f(c)|/2$ there is an interval around c such that

  $$\exists I_\delta(c) \subseteq f^{-1}(I_\epsilon(c)) \subseteq I(c)$$

Thus on this interval, image of f is in

$$I_\epsilon(c) = f(c) + \left(-\frac{|f(c)|}{2}, \frac{|f(c)|}{2}\right)$$

$\implies$ **f has the same sign as $f(c)$ on $I_\delta(c)$.**

, there is a $I_\epsilon(c) \subseteq (a,b)$ in which $f^*_c(x)$ has same sign as $f^*(c) = f'(c) > 0$.

- This means $f(x) - f(c)$ has same sign as $x - c$.

the derivative cannot be positive or negative or $\pm\infty$, otherwise it will be **strictly increasing** locally. So the derivative must be 0

- If $M, m \in \{a, b\}$, then

$$f(a) = f(b) \implies M = m$$

so the minima and maxima values are equal so the function must be a constant: $f(a) = f(b) = f(x)$ for all $x \in (a,b)$. A constant function is always differentiable and derivative is zero: $f'(x) = 0$ for all $x \in (a,b)$.

- From the proof of Rolle's theorem, we know more about the point c, as we may choose c to be a global maximum or a global minimum for a non-constant f.
- **If f is non-constant**, global max and min must be necessarily have different values and are on two distinct points M, m where

$$f(M) := \sup f[a,b] \neq f(m) := \inf f[a,b]$$

- In general **both** points would not be in the interior or have derivative zero by the example of $x(1-x)$ on $[0,1]$.

Example

$x(1-x)$ on $[0,1]$

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    left=-1; right=2;
    top=1; bottom=-1;
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x(1-x)
```

mean value theorems

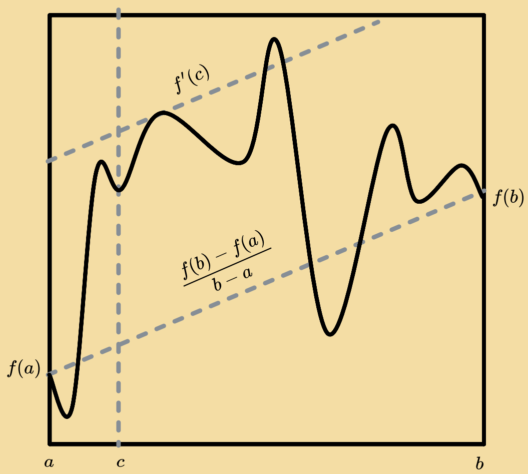
Even though the statement does not contain all the information from the proof, still the statement of Rolle's has consequences.

(Lagrange's mean value theorem) **Let a continuous function**

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b)**. Then there is a point** $c \in (a, b)$ **at which**

$$f(b) - f(a) = (b - a)f'(c)$$

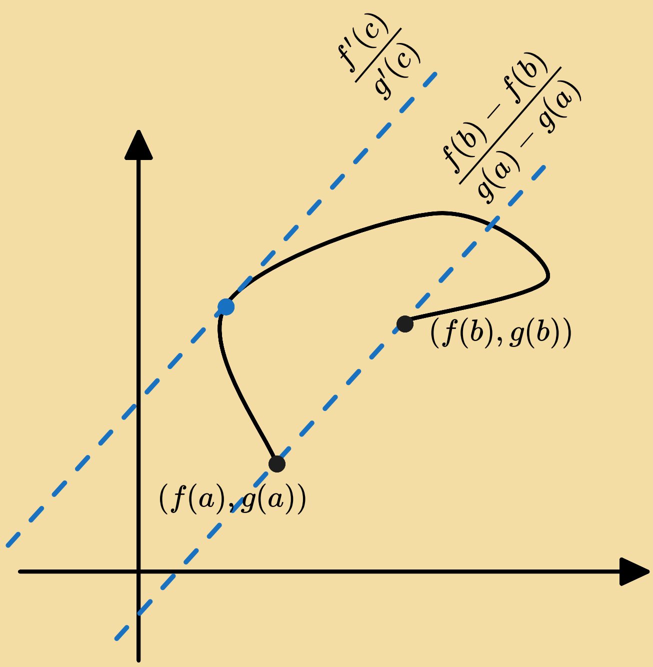


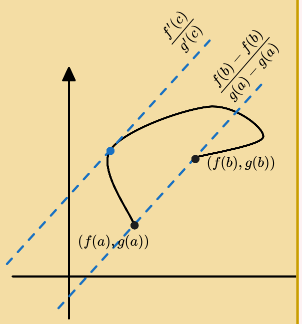
(Cauchy's mean value theorem) **Let two continuous**

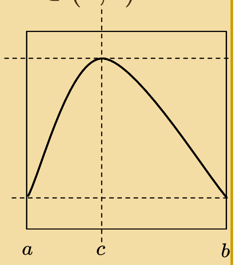
$$(f, g) : [a, b] \to \mathbb{R}^2$$

be differentiable on (a, b)**. Then there is a point** $c \in (a, b)$ **at which**

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$



		applying Rolles' on
Cauchy's mean value theorem	**(Cauchy's mean valu theorem) Let two continuous** $(f, g) : [a, b] \to \mathbb{R}^2$ **be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which** $(f(b) - f(a))g'(c) = (g$  The difference $h(t) := (f(b) - f(a))g(t)$ - on $t \in [a, b]$, Then h is continuous on $[a, b]$ and differentiable in (a, b) and $h(a) = f(b)g(a) - f(a)g(b$ Apply **(Rolle's theorem extended by Cauchy) Let a continuous** $f : [a, b] \to \mathbb{R}$ $f(a) = f(b)$ **such that** $f' : (a, b) \to \mathbb{R} \cup \{$	the difference $h(t) := (f(b) - f(a))g(t) - (g($ on $t \in [a, b]$, Then h is continuous on $[a, b]$ and differentiable in (a, b) and $h(a) = f(b)g(a) - f(a)g(b) =$

		applying Rolles' on
	exists. Then there is a point in the interior where the derivative is zero $\exists c \in (a, b)$ such th  • (Extremum value theorem for continuous $[a, b] \to \mathbb{R}$) Let $f : [a, b]$ - be continuous, then a global maxima *and* a minima exists $M, m \in [$ for f. By A continuous function	

		applying Rolles' on
	from a compact metric space to $\mathbb{R}$ has a global maxima *and* a global minima. - Image is a compact subset	

		applying Rolles' on
	of $\mathbb{R}$, so it closed and bounded, so its su and in is in th	

		applying Rolles' on
	eset. • The su and in are global maxima and minima. ∎ ∎	

		applying Rolles' on
	• Define M, m such that $f(M) = \sup f$ and $f(m) = \inf f$ • If one of M, m is in the interior (a, b) then $f'(m) = 0$ or $f'(M) = 0$ at that extremum value because of **(Derivative, if it exists, must be 0 at local extremas)** Let a function $f : ($ have a local maxima or min	

			applying Rolles' on
		ma at $c \in$. If f has an exte nde deri ativ at c then the deri ativ mus t be zero $f'(c$ The efor e, $\{\mathrm{loc}$ By	

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			applying Rolles' **on**
		the derivative cannot be positive or negative or $\pm\infty$, otherwise it will be **strictly increasing** locally. So the derivative must be 0	

		applying Rolles' on
	• If $M, m \in \{a,$, then $f(a) = f(b)$ so the minima and maxima values are equal so the function must be a constant: $f(a) = f(b)$ for all $x \in (a, b)$. A constant function is always differentiable and derivative is zero: $f'(x) = 0$ for all $x \in (a, b)$.	
Lagrange's mean value theorem		In the *generalized mean value theorem* we put $g(t) = t$ $h(t) := (f(a) - f(b))t - (b -$

		applying Rolles' on
		then $$h(a) = f(b)a - f(a)b = h(b)$$
	$$(f(b) - f(a))c = \frac{1}{2}(b^2 - a^2)f'$$	put $g(t) = \frac{t^2}{2}$ we have
for determinants	Let three continuous $$g_1, g_2, g_3 : [a, b] \to \mathbb{R}$$ be differentiable on (a, b). Then for $$D(x) := \det \begin{bmatrix} g_1(x) & g_2(x) & g \\ g_1(a) & g_2(a) & g \\ g_1(b) & g_2(b) & g \end{bmatrix}$$	We observe $$D'(x) = \det \begin{bmatrix} g_1'(x) & g_2'(x) & g \\ g_1(a) & g_2(a) & g \\ g_1(b) & g_2(b) & g \end{bmatrix}$$ and $$D(a) = 0 = D(b)$$ thus we apply *Rolle's mean value theorem* on D itself.

monotonicity

Corollary of mean value theorem: If

$$f : (a, b) \to \mathbb{R}$$

is differentiable, then for any proper subinterval $(x_1, x_2) \subset (a, b)$ by mean value theorem we have

$$\exists x \in (x_1, x_2) : f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$$

Thus

Hypothesis on f'	Consequence of Corollary of mean value theorem: If $$f : (a, b) \to \mathbb{R}$$ is differentiable, then for any proper subinterval $(x_1, x_2) \subset (a, b)$ by mean value theorem we have $$\exists x \in (x_1, x_2) : f(x_2) - f(x_1) = (x_2 -$$
$f' \geq 0$ on (a, b)	$$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \geq 0$$ thus f is **increasing** on (a, b)
$f' = 0$ on (a, b)	$$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) = 0$$ thus f is **constant** on (a, b)
$f' \leq 0$ on (a, b)	$$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \leq 0$$ thus f is **decreasing** on (a, b)
$f' < 0$ on (a, b)	$$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) < 0$$ thus f is **strictly decreasing** on (a, b)
$f' > 0$ on (a, b)	$$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) > 0$$ thus f is **strictly increasing** on (a, b)
$f' = \alpha$ is a constant on (a, b) or $$f'' = 0$$ on (a, b)	$f(x) - \alpha x$ has zero derivative, thus f is **linear** $$f(x) \equiv f'(0)x + f(0)$$

Hypothesis on $$f''$$ or higher derivatives	Consequence of
	Corollary of mean value theorem: If $$f:(a,b)\to\mathbb{R}$$ is differentiable, then for any proper subinterval $(x_1,x_2)\subset(a,b)$ by mean value theorem we have $$\exists x\in(x_1,x_2): f(x_2)-f(x_1)=(x_2 -$$
$f'=\alpha$ is a constant on (a,b) or $$f''=0$$ on (a,b)	$f(x)-\alpha x$ has zero derivative, thus f is **linear** $$f(x)\equiv f'(0)x+f(0)$$
$$f''\geq 0$$ on (a,b)	$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+g(x)}{2}$$ thus f is **convex** on (a,b)
$$f''\leq 0$$ on (a,b)	$$f\left(\frac{x+y}{2}\right)\geq\frac{f(x)+g(x)}{2}$$ thus f is **concave** on (a,b)
$$f^{(n)}=0$$ on (a,b)	f is a **polynomial** $$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k$$

intermediate value property of derivatives

Let

$$f:[a,b]\to\mathbb{R}$$

be differentiable and $\lambda\in(f'(a),f'(b))$. **Then there is a** $c\in[a,b]$ **such that**

$$f'(c)=\lambda$$

bounded derivative and uniform continuity

Let

$$f : U(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$$

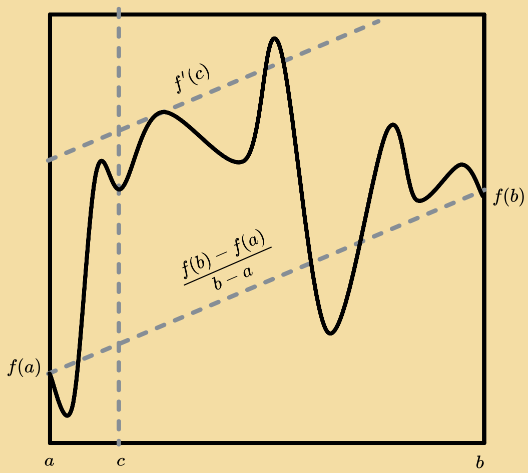
have a bounded derivative $|f'| \leq M$ for some $M \geq 0$. Then for every $x, y \in U$ by

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(c)$$



there is a $c \in (x, y)$ such that

$$\begin{aligned} f(y) - f(x) &= (y - x) f'(c) \\ |f(y) - f(x)| &= |y - x| \, |f'(c)| \\ &\leq M \, |y - x| \end{aligned}$$

So for any $\epsilon > 0$ choosing $\delta := \epsilon / M$ we have

$$x, y \in U, |y - x| \leq \delta = \frac{\epsilon}{M} \implies |f(y) - f(x)| \leq M \frac{\epsilon}{M} = \epsilon$$

Hence, f is **uniformly continuous** on U.

$\mathcal{C}^1 \implies$ locally Lipschitz

Let

$$f : [a, b] \to \mathbb{R}$$

be differentiable and

$$f' : [a, b] \to \mathbb{R}$$

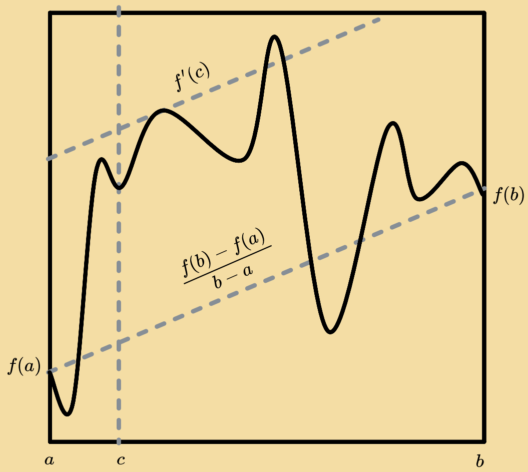
be continuous. Then f' is bounded, say $|f'| \leq M$. Then for any $x, y \in [a, b]$ by

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(c)$$



we have

$$\exists c \in (a, b) : \left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq M$$

Hence, f is **Lipshitz continuous** on $[a, b]$.

bounded derivative $\iff$ Lipshitz

Let

$$f : U(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$$

be differentiable.

- Then f being M-Lipshitz implies

  $$\left| \frac{f(y) - f(y+h)}{h} \right| \leq M \implies |f'(y)| \leq M$$

  so f has a bounded derivative on U.
- Let f have a bounded derivative on U, $|f'| \leq M$. Then for any $x, y \in U$, by

(Lagrange's mean value theorem) Let a continuous function

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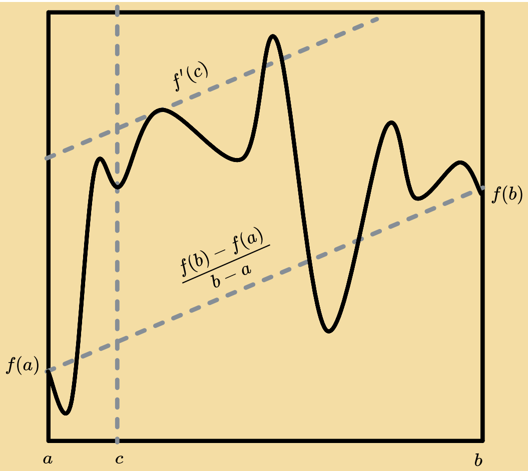
be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(c)$$



we have

$$\exists c \in (x,y) : \left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \le M$$

Thus, f is M-**Lipschitz**.

differentiable functions on the circle

Bug

If we have a differentiable function on the circle S^1, that is equivalent to a continuous

$$f : [a,b] \to \mathbb{R}$$
$$f(a) = f(b)$$

which is differentiable on (a,b) and the left hand derivative at $b =$ right hand derivative at a.
Then such a non-constant f has distinct maxima and minima

If we have a differentiable non-constant function on S^1 then we have

$$f(M) := \sup f[a,b] \neq f(m) := \inf f[a,b]$$
$$f'(M) = 0 = f'(m)$$

mean value for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$

- Let

$$f : U \text{ (open)} \subseteq \mathbb{R}^n \to \mathbb{R}$$

be differentiable

$$\mathrm{d}f : U \to \mathbb{R}^{n*}$$

- For any $x, y \in U$ such that the line segment $\overline{xy} \subseteq U$ we have

$$\begin{aligned} f \circ \overline{xy} : [0,1] &\to \mathbb{R} \\ t &\mapsto f((1-t)x + ty) \end{aligned}$$

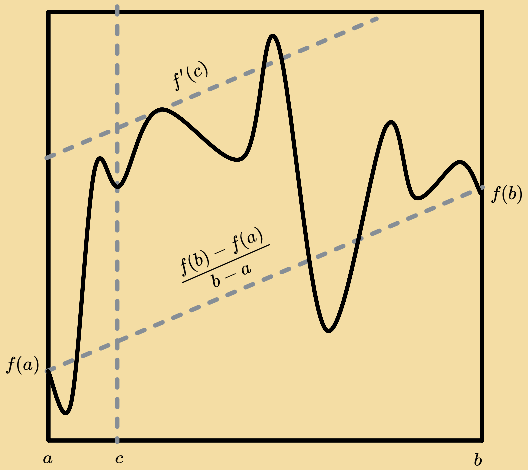
- By

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(c)$$



there is a $c \in (0, 1)$ such that

$$\begin{aligned} f(y) - f(x) &= \frac{\mathrm{d}}{\mathrm{d}t} f((1-t)x + ty)\Big|_{t=c} \\ &= \mathrm{d}_{(1-c)x+cy} f(y - x) \end{aligned}$$

- This implies

$$|f(y) - f(x)| \le \left\|\mathrm{d}_{(1-c)x+cy} f\right\|_{\mathbb{R}^{n*}} |y - x|$$

for the operator norm on $\mathbb{R}^{n*}$.

Let

$$f : U \text{ (open, convex)} \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

be differentiable

$$\mathfrak{D} f : U \to \mathrm{EndVec}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$$

such that the operator norm of the derivative on U is bounded by M

$$\forall x \in U, \|\mathfrak{D}_x f\|_{n \to m} \le M$$

Then f is M-Lipshitz on U

$$\forall x, y \in U, |f(y) - f(x)| \leq M\, |y - x|$$

Example

bounded derivative but non-Lipshitz

The operator norm of the derivative

$$\|\mathrm{d}\theta\| = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} = \frac{1}{\sqrt{x^2 + y^2}}$$

is bounded on complement of the unit disk

$$\|\mathrm{d}\theta\| \leq 1 \text{ on } \left(\mathbb{R}^2 \setminus (-\infty, 0)\right) \setminus \overline{D^2}$$

However,

$$\frac{|\theta(-2, \epsilon) - \theta(-2, -\epsilon)|}{|(-2, \epsilon) - (-2, -\epsilon)|} \xrightarrow{\epsilon \to 0} \frac{2\pi}{2\epsilon}$$

Therefore,

$$|\theta(y) - \theta(x)| \leq \underbrace{\|\mathrm{d}\theta\|}_{\leq 1} |y - x|$$

does not hold.

[1]

1. https://math.stackexchange.com/questions/704108/a-generalization-of-the-mean-value-theorem ↩

Current note has 1 direct children and 1 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - mapping
      - mean value Mean value property of functions
        - iterative Taylor's theorem

And it has 4 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - mapping
      - balls Mapping balls
      - inverse Inverse of differentiable maps
      - mean value Mean value property of functions
      - seqf Sequence of functions on $\mathbb{R}^d$