Info

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Monotone functions on $\mathbb{R}$

Definition. Motonone functions on $\mathbb{R}$

Let $f : I\,(\text{open interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then

- f is **monotonically increasing** on I if for each $x, y \in I$ $x \geq y$ implies $f(x) \geq f(y)$
- f is **strictly increasing** on I if for each $x, y \in I$ $x > y$ implies $f(x) > f(y)$
  - $\iff$ for $c \in I$, $f(x) - f(c)$ has the same sign as $x - c$ for all $x \in I$

differentiable maps and monotonicity

Let $f : I\,(\text{open interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable. Then f is strictly monotonic $\iff$ $Z(f')$ does not contain an open interval/has empty interior/every stationary point is isolated and f' does not change sign on $I \setminus Z(f')$.

($\implies$)

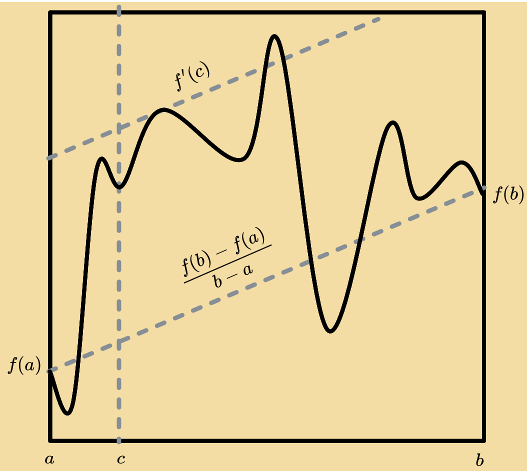
- If $Z(f')$ contains an open interval, then by

  (Lagrange's mean value theorem) Let a continuous function

  $$f : [a, b] \to \mathbb{R}$$

  be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

  $$f(b) - f(a) = (b - a) f'(c)$$



f is constant on that interval. Hence, it is not strictly monotonic.

- Let f is strictly decreasing. Then by

(Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ and

$$f'(c) > 0 \text{ or } + \infty$$

then there is a $I(c) \subseteq (a, b)$ in which $f(x) - f(c)$ has same sign as $x - c$, that is, f is strictly increasing on $I(c)$.

- If $f'(c) > 0$, then by
  - Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.
    - Define

$$f_c^*(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

      - f^* is continuous in (a, b).
    - So we can write f as

$$f(x) - f(c) = (x - c) f_c^*(x)$$

      for an unique continuous f_c^*.

  we have

$$f(x) - f(c) = (x - c) f_c^*(x)$$

  for a continuous f_c^*
- By

(Continuity preserves sign locally): Let $I(c)$ be an interval containing c and

$$f : I(c) \subset \mathbb{R} \to \mathbb{R}$$

be continuous at $c \in I(c), f(c) \neq 0$. Then for $\epsilon := |f(c)|/2$ there is an interval around c such that

$$\exists I_\delta(c) \subseteq f^{-1}(I_\epsilon(c)) \subseteq I(c)$$

Thus on this interval, image of f is in

$$I_\epsilon(c) = f(c) + \left(-\frac{|f(c)|}{2}, \frac{|f(c)|}{2}\right)$$

$\implies$ f has the same sign as $f(c)$ on $I_\delta(c)$.

, there is a $I_\epsilon(c) \subseteq (a, b)$ in which $f_c^*(x)$ has same sign as $f^*(c) = f'(c) > 0$.

- This means $f(x) - f(c)$ has same sign as $x - c$.

∎

$f' < 0$ on $I \setminus Z(f')$.

($\Longleftarrow$) Let $f' \geq 0$. Then by

Hypothesis on f'	Consequence of **Corollary of mean value theorem:** If $$f : (a, b) \to \mathbb{R}$$ is differentiable, then for any proper subinterval $(x_1, x_2) \subset (a, b)$ by mean value theorem we have $$\exists x \in (x_1, x_2) : f(x_2) - f(x_1) = (x$$
$f' \geq 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \geq 0$ thus f is increasing on (a, b)
$f' = 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) = 0$ thus f is constant on (a, b)
$f' \leq 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \leq 0$ thus f is decreasing on (a, b)
$f' < 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) < 0$ thus f is strictly decreasing on (a, b)

$f' > 0$ on (a,b)	$\forall x_1 < x_2 \in (a,b), f(x_1) - f(x_2) > 0$ thus f is **strictly increasing** on (a,b)
$f' = \alpha$ is a constant on (a,b) or $$f'' = 0$$ on (a,b)	$f(x) - \alpha x$ has zero derivative, thus f is **linear** $$f(x) \equiv f'(0)x + f(0)$$

we conclude f is increasing on I.

- If there are $x, y \in I$ such that $f(x) = f(y)$, then

$$\begin{aligned} & f \equiv f(x) \text{ on } [x.y] \\ \implies & f' \equiv 0 \text{ on } [x,y] \end{aligned}$$

- So if $Z(f')$ are isolated, we conclude that f must be **strictly increasing**.

Let $A \subseteq [0,1]$ be a compact subset with empty interior. Then there exists a $f \in \mathcal{C}^1[0,1]$ such that $Z(f') = A$.

Let the connected components of $[0,1] \setminus A$ be

$$I_1, \dots$$

- Then

$$I_n \cap (0,1) = (a_n, b_n)$$

- ...

[1]

Current note has 1 direct children and 1 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - monotone Monotone functions on $\mathbb{R}$
      - asymp Asymptotics of monotonically increasing functions $[0,\infty] \to [0,\infty]$

And it has 36 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - circle packing Circle packing on $\mathbb{R}^2$
    - circle packing to Riemann map Circle packing converges to the Riemann biholomorphism

- Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
- Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
- cont Continuous functions on $\mathbb{R}^d$
- cube dyadic Dyadic cubes
- curves Curves
- derivative Differentiable functions
- forms Differential forms on $\mathbb{R}^n$
- Fourier-Wigner Fourier-Wigner transform
- harmonic composed conformal Harmonic functions composed with conformal maps
- Hilbert Hilbert transform
- hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
- Hol sets Holomorphic subsets of $\mathbb{C}^n$
- hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
- hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
- KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

- Laplace Laplace operator on $\mathbb{R}^n$
- Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- met density Metric density of subsets of $\mathbb{R}^n$
- Mobius n-sphere Mobius endomorphisms
- monotone Monotone functions on $\mathbb{R}$
- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$
- Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$

- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$

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1. https://math.stackexchange.com/a/1845973/1290493 ↩