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Quasiconformal maps on $\mathbb{C}$

Definition. Quasiconformal maps between open subsets of $\mathbb{C}$

Let $U, V \subseteq \mathbb{C}$ and $K \geq 1$. Then a homeomorphism

$$f : U \to V$$

is ***K*-quasiconformal** if its distributional partial derivatives are in $L^2_{\text{loc}}(U)$ and satisfy

$$\left| \frac{\overset{\int}{\partial f}}{\partial \bar{z}} \right| \leq \frac{K-1}{K+1} \left| \frac{\overset{\int}{\partial f}}{\partial z} \right| \text{aeon } U$$

- The set of all such maps

$$\mathcal{Q}_K(U,V) \subseteq H^1_{\text{loc}} \cap \mathcal{C}(U)$$

  is contained in the set of all quasiconformal maps

$$\mathcal{Q}(U,V) := \bigcup_{K \geq 1} \mathcal{Q}_K(U,V)$$

- We also have the set of all quasiconformal mapping into $\mathbb{C}$ $\mathcal{Q}(U) := \bigcup_{V \subseteq \mathbb{C}} \mathcal{Q}(U,V)$ and $\mathcal{Q}_K(U) := \bigcup_{V \subseteq \mathbb{C}} \mathcal{Q}_K(U,V)$.
- We have the **Beltrami coefficient** of a quasiconfomal map

$$\mu : \mathcal{Q}(U) \to L^\infty(U)$$
$$f \mapsto \mu(f)(z) := \frac{\overset{\int}{\partial f}}{\partial \bar{z}} \left(\frac{\overset{\int}{\partial f}}{\partial z} \right)^{-1}$$

  and the **eccentricity**

$$K : \mathscr{Q}(U) \to L^\infty(U)$$
$$f \mapsto K(f)(z) := \frac{1+|\mu(f)(z)|}{1-|\mu(f)(z)|}$$

The smallest K such that f is K-quasiconformal is called the **quasiconformal constant** $\|K(f)\|_\infty$ of f.

(Weyl's lemma)

$$\mathscr{Q}_1(U,V) \subset \mathscr{O}(U,V)$$

The map

$$\mu : \mathscr{Q}(U) \to L^\infty(U)$$
$$f \mapsto \mu(f)(z) := \frac{\partial f}{\partial \bar{z}} \left(\frac{\partial f}{\partial z} \right)^{-1}$$

is surjective and the fiber containing $f \in \mathscr{Q}(U)$ is precisely left composition of f by injective holomorphic mappings

$$\mu^{-1}(\mu(f)) = \{\varphi \circ f | \varphi \in \mathscr{Q}_1(f(U))\}$$

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extension to boundary

$$\mathscr{O}_K(H^2_{\mathrm{U}}, H^2_{\mathrm{U}}) \hookrightarrow \mathcal{C}(\mathbb{R}\boldsymbol{P}^1, \mathbb{R}\boldsymbol{P}^1)$$

and

$$\mathscr{O}_K(H^2_{\mathrm{U}}, H^2_{\mathrm{U}}) \hookrightarrow \mathscr{O}_K(\mathbb{C}\boldsymbol{P}^1, \mathbb{C}\boldsymbol{P}^1)$$
$$f \mapsto \begin{cases} f \\ f(\bar{z}) = \overline{f(z)} \end{cases}$$

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- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$

- quasiconformal
  - 2 Quasiconformal maps on $\mathbb{C}$

And it has 2 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - quasiconformal
      - 2 Quasiconformal maps on $\mathbb{C}$
      - n Quasiconformal maps on S^n

Theorem 4.6.1 (The mapping theorem)

1. *Let $U \subset \mathbb{C}$ be open and let $\mu \in L^\infty(U)$ satisfy $\|\mu\|_\infty < 1$. Then there exists a quasiconformal mapping $f : U \to \mathbb{C}$ satisfying the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}. \qquad 4.6.1$$

2. *If g is another quasiconformal solution to equation 4.6.1, then there exists an injective analytic function $\varphi : f(U) \to \mathbb{C}$ such that $g = \varphi \circ f$.*

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