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Quasiconformal maps on $S^n \cong \overline{\mathbb{R}^n}^{\infty}$

Definition. Quasiconformal maps between open subsets of $\mathbb{R}^n$

Consider a homeomorphism $f : U_1 \subseteq \mathbb{R}^n \to U_2 \subseteq \mathbb{R}^n$ between two open subsets U_1, U_2. The **linear dilation** of f at $x \in U_1$ is

$$\begin{aligned} H_x(f) &:= \limsup_{r\to 0} \sup \left\{ \frac{|f(x)-f(y)|}{|f(x)-f(z)|} \middle| y, z \in U_1 \cap S_r(x) \right\} \\ &= \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)| | y \in U_1 \cap B_x(r)\}}{\inf\{|f(x)-f(y)| | y \in U_1 \setminus B_x(r)\}} \end{aligned}$$

The homeomorphism f is **quasiconformal** if $\sup_{x\in U_1} H_x(f) < \infty$.

$$\|\mathfrak{D}_x f\| \leq M$$

$$\sup_{y,z\in S_r(x)} \frac{|f(x)-f(y)|}{|f(x)-f(z)|} = \frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}}$$

$$\sup_{y\in S_r(x)} \leq \sup_{y\in B_r(x)}$$

and

$$\inf_{y\notin B_r(x)} \leq \inf_{y\in S_r(x)}$$

$$\begin{aligned} |f(x)-f(y)| &\leq M(r_k) \\ \frac{1}{|f(x)-f(z_k)|} &\leq \frac{1}{m(s_k)} \end{aligned}$$

$$\frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}} \leq \frac{\sup_{y\in B_r(x)}}{\inf_{z\notin B_r(x)}} \leq \limsup_{r\to 0} \frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}}$$

3 Quasisymmetric mappings

By Theorem 2.1 we know that quasiconformality implies the uniform local boundedness of $H_f(x,r)$. We introduce the equivalent concept of quasisymmetry that turns out to be very useful.

Let X and Y be metric spaces and let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $f : X \to Y$ is η-quasisymmetric (η-qs), if

$$\frac{|f(a)-f(x)|}{|f(b)-f(x)|} \leq \eta\left(\frac{|a-x|}{|b-x|}\right)$$

for all $a \neq x \neq b$.

3.1 Remark. If f is η-quasisymmetric, then

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)} \leq \eta(1).$$

So, quasisymmetric mappings are quasiconformal.

We next prove that quasiconformal mappings are locally quasisymmetric.

3.2 Theorem. Let $f : B(x_0, 3r_0) \to \Omega' \subset \mathbb{R}^n$ be a homeomorphism such that $H_f(x,r) \leq H$ for all $x \in B(x_0,r_0)$ and $0 < r < 2r_0$. Then $f|_{B(x_0,r_0)}$ is η-quasisymmetric, where η depends only on n and H.

4.1 Proposition. Let $f : \Omega \to \Omega'$ be a homeomorphism. Then

$$\mu_f'(x) = \lim_{r\to 0} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}$$

exists almost everywhere in Ω, belongs to $L^1_{\text{loc}}(\Omega)$ and

$$\int_E \mu_f'(x)\,dx \leq |f(E)|$$

for each Borel set $E \subset \Omega$, with equality whenever $|A| = 0$ implies $|f(A)| = 0$.

Let $f : \Omega_1 \subseteq \mathbb{R}^n \to \Omega_2 \subseteq \mathbb{R}^n$ be a homeomorphism. The function

$$L_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{r}$$

is Borel measurable and

$$\mu_f'(x) \leq L_f(x)^n \leq H_f(x)^n \mu_f'(x)$$

for almost every $x \in \Omega_1$. In particular, $L_f \in L^n_{\mathrm{loc}}(\Omega_1)$ then f is quasiconformal.

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Consider the closed subsets

$$A_n := \left\{ x \in E \middle| \forall 0 < |h| < \frac{d(E,\partial\Omega)}{n}, \frac{|f(x+h) - f(x)|}{|h|} \leq t - \frac{1}{n} \right\}$$

then for compact E

$$\{L_f < t\} = \bigcup_{n\geq 1} A_n$$

4.10 Lemma. (Reverse Hölder Inequality) Let f be η-quasisymmetric on $2B \subset \mathbb{R}^n$. Then

$$\left(\fint_B L_f^n\right)^{1/n} \leq C(n,\eta) \fint_B L_f.$$

4.12 Corollary. Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$. Then $|f(E)| = 0$, if and only if $|E| = 0$. In particular,

$$|f(E)| = \int_E \mu_f' \, dx,$$

for Borel (and all Lebesgue measurable) sets E, and f maps Lebesgue measurable sets to Lebesgue measurable sets. Moreover, $\mu_f'(x) > 0$ almost everywhere.

- Let $m(E) = 0$.
  - Given $\epsilon > 0$ consider an open set V with $E \subset V \subset U$ and $m(V) < \epsilon$
  - For each $x \in E$ we pick

$$\overline{B_{r(x)}(x)}$$

  such that $\overline{B_{15r(x)}}(x) \subset V$.

$$f(E) \subset f\left(\bigcup_j 5\overline{B_j}\right)$$

- $$\left| f\left(\bigcup_i 5\overline{B_j}\right)\right| \leq \sum_j \left| f\left(5\overline{B_j}\right)\right| \leq \eta(5)c(n)\sum_j L_f(x_j, r_j)^n$$
- ...
- Letting $\epsilon \to 0$ we conclude $m(f(E)) = 0$.

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- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - quasiconformal
      - n Quasiconformal maps on S^n

And it has 2 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - quasiconformal
      - 2 Quasiconformal maps on $\mathbb{C}$
      - n Quasiconformal maps on S^n

4.2 The maximal streching

Set

$$L_f(x) = \limsup_{r\to 0} \frac{L_f(x,r)}{r}.$$

4.4 Lemma. Let $f : \Omega \to \Omega'$ be a homeomorphism. The function L_f is Borel measurable and

$$\mu'_f(x) \leq L_f(x)^n \leq H_f(x)^n \mu'_f(x)$$

for almost every $x \in \Omega$. In particular, $L_f \in L^n_{\text{loc}}(\Omega)$ when f is quasiconformal.

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