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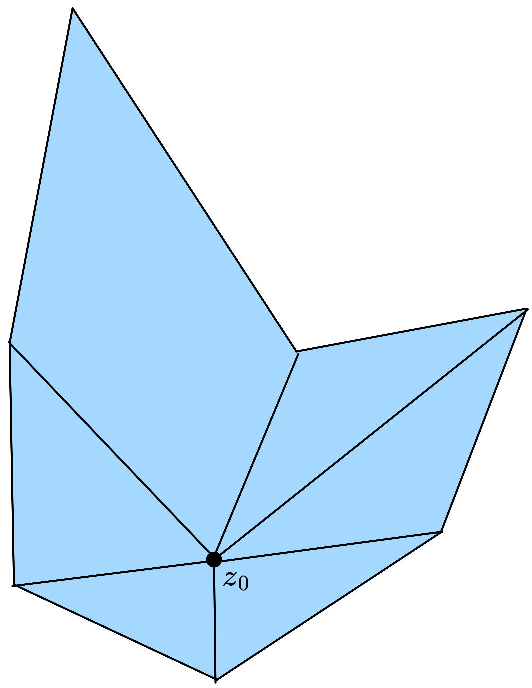
# Star-shaped subsets of $\mathbb{R}^n$

**Definition. Star-shaped subsets of $\mathbb{R}^n$**

A subset $S \subseteq \mathbb{R}^n$ is called **star-shaped** if there exists a point $z_0 \in S$ such that the line segment between $z_0$ and any point $z \in S$

$$z_0 + t(z_0 - z),\ t \in [0, 1]$$

is contained in $S$.




Then $z_0$ is called a **center** of $S$.

## examples

- $\mathbb{R}^n$

- convex sets

## they are contractible

**Star-shaped subsets of $\mathbb{R}^n$ are contractible.**

Take a center $q$ of the star-shaped subset, then $\mathsf{Id}_U$ is homotopic to the constant map $c_q : x \in U \mapsto q$ by the obvious and natural *straight-line* homotopy:

$$H(x,t) := q + t(x-q)$$

Thus the inclusion $\{q\} \to U$ is a homotopy equivalence.

## cohomology of differential forms

**For any star-shaped subset $U$ of $\mathbb{R}^n$, its de Rham cohomology are**

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k>0 \end{cases}$$

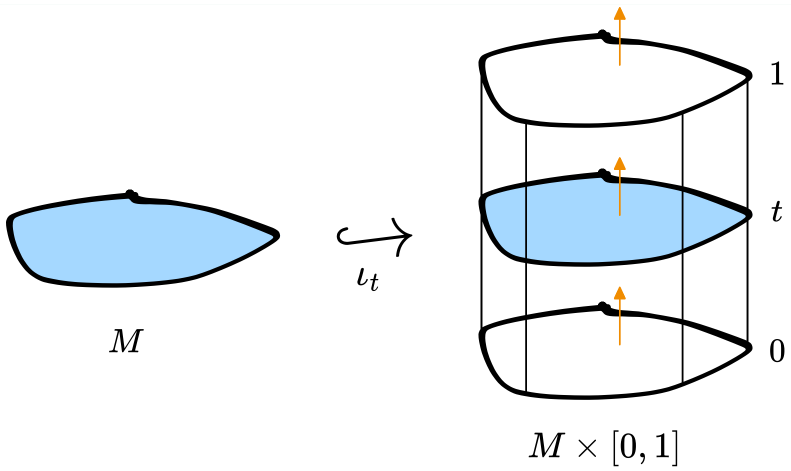
Let $U \subseteq \mathbb{R}^n$ be a star-shaped open set with center at $0 \in \mathbb{R}^n$. Consider differential forms on $U \subseteq \mathbb{R}^{n+1}$ and the homotopy operator

Definition. **Homotopy operator from forms on $M \times I$ to forms on $M$**

Let $M$ be a smooth manifold with or without boundary and consider the map

$$\begin{aligned} \iota_t : M &\to M \times [0,1] \\ x &\mapsto (x,t) \end{aligned}$$

for a fixed $t \in [0,1]$.




Let $(-, s) \in M \times [0,1]$ be the coordinate. We define

$$h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$$
$$\omega \mapsto h\omega$$
$$(h\omega)_q := \int_{t\in[0,1]} \iota_t^*(\iota_{\frac{\partial}{\partial s}}(\omega_{q,t}))$$
$$(h\omega)_q(-) = \int_{t\in[0,1]} \omega(\frac{\partial}{\partial s}, \mathrm{d}\iota_t(-))$$

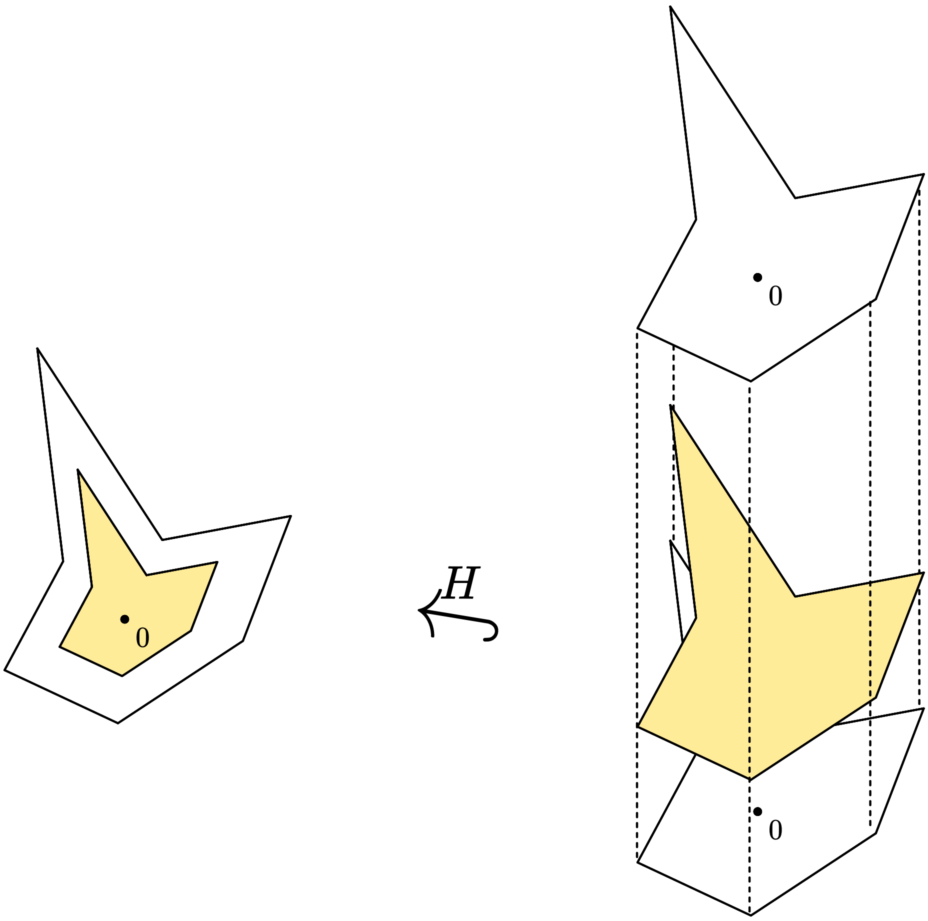
which satisfies

Proposition:

$$(\iota_1^* - \iota_0^*)(\omega) = h(\mathrm{d}\omega) + \mathrm{d}(h\omega)$$

Finally with

$$H : U \times [0,1] \to U$$
$$H(x,t) = tx$$




let

$$J := h \circ H^* : \Omega^k(U) \to \Omega^{k-1}(U)$$

then we have

$$H^*(\alpha) = \alpha(\mathrm{d}H(-), \mathrm{d}H(-), \ldots)$$
$$J(\alpha) = h(H^*(\alpha)) = \int_{t\in[0,1]} \alpha(\mathrm{d}H(\frac{\partial}{\partial s}), \underbrace{\mathrm{d}H(\mathrm{d}\iota_t(-))}_{t(-)}, \ldots)$$
$$(J\alpha)_q(w_1, \ldots, w_k) = \int_{t\in[0,1]} \alpha_{tq}(q, tw_1, tw_2, \ldots, tw_k)$$
$$(\underbrace{H_1^*}_{\mathsf{Id}} - \underbrace{H_0^*}_{0})(\alpha) = J\mathrm{d}\alpha + \mathrm{d}J\alpha$$

Thus we have

$$\alpha = J\mathrm{d}\alpha + \mathrm{d}J\alpha$$

which for $\mathrm{d}\alpha = 0$ gives

$$\alpha = \mathrm{d}(J\alpha)$$

thus we have a anti-derivative for $\alpha$ implying it is exact.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - star shaped Star-shaped subsets of $\mathbb{R}^n$

And it has 36 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
        - circle packing Circle packing on $\mathbb{R}^2$
        - circle packing to Riemann map Circle packing converges to the Riemann biholomorphism
        - Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
        - Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
        - cont Continuous functions on $\mathbb{R}^d$
        - cube dyadic Dyadic cubes
        - curves Curves
        - derivative Differentiable functions
        - forms Differential forms on $\mathbb{R}^n$
        - Fourier-Wigner Fourier-Wigner transform
        - harmonic composed conformal Harmonic functions composed with conformal maps
        - Hilbert Hilbert transform
        - hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
        - Hol sets Holomorphic subsets of $\mathbb{C}^n$
        - hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
        - hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
        - KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

        - Laplace Laplace operator on $\mathbb{R}^n$
        - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
        - Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
        - met density Metric density of subsets of $\mathbb{R}^n$
        - Mobius n-sphere Mobius endomorphisms
        - monotone Monotone functions on $\mathbb{R}$

- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$
- Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$