ⓘ Info

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# Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the *unit disk* and their boundary value on the *unit circle*

## when the boundary value is continuous on the unit circle

## when the boundary value is $L^2$ on the unit circle

**Proposition:**

$$\begin{aligned} P : l^2(\mathbb{N}, \mathbb{C}) &\to \mathit{Hardy}^2(D) \subset \mathcal{H}(D) \\ (a_n) &\mapsto \sum_{n \geq 0} a_n z^n \end{aligned}$$

$$F(z) := \sum_{n \geq 0} a_n z^n$$

For any $r < 1$

$$F(re^{i\theta}) = \sum_{n \geq 0} a_n r^n e^{i\pi n\theta}$$

where $a_n r^n$ is the Fourier coefficient of $f(\theta) := F(re^{i\theta})$ given by

$$a_n r^n = \frac{1}{2\pi} \int_{\theta \in [-\pi, \pi]} F(re^{i\theta}) e^{-in\theta}$$

and the integral vanishes when $n < 0$.

$$\alpha_n = \begin{cases} a_0 & n = 0 \\ \frac{1}{2} a_n & n > 0 \\ \frac{1}{2} \bar{a}_{-n} & n < 0 \end{cases}$$

$$\begin{tikzcd}
 & \begin{matrix} l^2(\mathbb{N}, \mathbb{C}) \\ (a_n)_{n \geq 0} \end{matrix} & \\
\begin{matrix} \overline{\mathbb{C}\{e^{inx} \mid n \in \mathbb{N}\}} < L^2([-\pi, \pi], \mathbb{C}) \\ f(\theta) = \sum_{n \geq 0} a_n e^{in\theta} \end{matrix} \arrow[ur, "a_n = \frac{1}{2\pi} \int_{[\pi, \pi]} f(\theta) e^{-in\theta}"] \arrow[rr, bend right, "f \mapsto F(z) = \frac{1}{2\pi i} \int_{\theta \in [-\pi, \pi]} \frac{f(e^{i\theta})}{e^{i\theta} - z}"'] \arrow[d, "\Re ?"] & & \begin{matrix} \mathit{Hardy}^2(D) < \mathcal{H}(D) \\ F(z) = \sum_{n \geq 0} a_n z^n \end{matrix} \arrow[ul, "a_n r^n = \frac{1}{2\pi} \int_{\theta \in [-\pi, \pi]} F(re^{i\theta}) e^{-in\theta}"'] \arrow[ll, bend right, "F \mapsto f = \lim^{L^2}_{r \to 1-} F(re^{i\bullet})"'] \arrow[d, "\Re"] \\
L^2([-\pi, \pi], \mathbb{R}) \arrow[rr, bend right, "f \mapsto u(re^{i\theta}) = \frac{1}{2\pi} \int_{t \in [-\pi, \pi]} \mathrm{Poisson}_D(r, \theta - t) f(e^{it})"'] \arrow[dr] & & \begin{matrix} \mathit{harmonic}^2(D) \\ u(re^{i\theta}) = \sum_{n \in \mathbb{Z}} \alpha_n r^n e^{2\pi in\theta} \end{matrix} \arrow[ll, bend right, "u \mapsto f = \lim^{L^2}_{r \to 1-} u(re^{i\bullet})"'] \arrow[dl, "\alpha_n = r^{-|n|} \int_{\theta \in [-\pi, \pi]} u(re^{i\theta}) e^{-in\theta}"] \\
 & \begin{matrix} l^2(\mathbb{Z}, \mathbb{R}) \\ (\alpha_n)_{n \in \mathbb{Z}} \end{matrix} &
\end{tikzcd}$$

$$\Re \int_{S^1} \frac{f(e^{i\theta})}{e^{i\theta} - z} \mathrm{d}\theta$$

**Proposition:**

$$\sum_{n \geq 0} \hat{f}(n) z^n = \frac{1}{2\pi i} \int_{\theta \in [-\pi, \pi]} \frac{f(e^{i\theta})}{e^{i\theta} - z}$$

$$\begin{aligned} u(re^{it}) &= \Re \left( \frac{1}{2\pi i} \int_{\theta \in [-\pi, \pi]} \frac{f(e^{i\theta})}{e^{i\theta} - re^{it}} \right) \\ &= \frac{1}{2\pi} \int \Re \left( \frac{-i}{e^{i\theta} - re^{it}} \right) f(e^{i\theta}) \\ &= \frac{1}{2\pi} \int \Re \left( \frac{-ie^{-i\theta}}{1 - re^{i(t-\theta)}} \right) f(e^{i\theta}) \end{aligned}$$

and here

$$\begin{aligned} \frac{1}{2\pi} \Re \left( \frac{-ie^{-i\theta}}{1 - re^{i(t-\theta)}} \right) &= \frac{1}{2\pi} \Re \left( \frac{-ie^{-i\theta}(1 - re^{-i(t-\theta)})}{(1 - re^{i(t-\theta)})(1 - re^{-i(t-\theta)})} \right) \\ &= \frac{1}{2\pi} \Re \left( \frac{i}{1 + r^2 -} \right) \end{aligned}$$

The diagram of $\mathbb{C}$-vector space endomorphisms of *Fourier* (left), *Fourier as well as Cauchy*
(top), *Poisson* (bottom)

$$
\begin{tikzcd}
 & \begin{array}{c} l^2(\mathbb{N},\mathbb{C}) \\ (a_n)_{n\geq 0} \end{array} & \\
\begin{array}{c} \overline{\mathbb{C}\{e^{inx} \mid n\in\mathbb{N}\}} < L^2([-\pi,\pi],\mathbb{C}) \\ f(\theta)=\sum_{n\geq 0} a_n e^{in\theta} \end{array} \arrow[ur, "a_n=\frac{1}{2\pi}\int_{[\pi,\pi]} f(\theta)e^{-in\theta}"] \arrow[rr, bend right, "f\mapsto F(z)=\frac{1}{2\pi i}\int_{\theta\in[-\pi,\pi]}\frac{f(e^{i\theta})}{e^{i\theta}-z}"'] \arrow[d, "\Re?"] & & \begin{array}{c} Hardy^2(D) < \mathcal{H}(D) \\ F(z)=\sum_{n\geq 0} a_n z^n \end{array} \arrow[ul, "a_n r^n=\frac{1}{2\pi}\int_{\theta\in[-\pi,\pi]} F(re^{i\theta})e^{-in\theta}"'] \arrow[ll, bend right, "F\mapsto f=\lim^{L^2}_{r\to 1-} F(re^{i\bullet})"'] \arrow[d, "\Re"] \\
L^2([-\pi,\pi],\mathbb{R}) \arrow[rr, bend right, "f\mapsto u(re^{i\theta})=\frac{1}{2\pi}\int_{t\in[-\pi,\pi]}\mathrm{Poisson}_D(r,\theta-t)f(e^{it})"'] \arrow[dr] & & \begin{array}{c} harmonic^2(D) \\ u(re^{i\theta})=\sum_{n\in\mathbb{Z}}\alpha_n r^n e^{2\pi in\theta} \end{array} \arrow[ll, bend right, "u\mapsto f=\lim^{L^2}_{r\to 1-} u(re^{i\bullet})"'] \arrow[dl, "\alpha_n=r^{-|n|}\int_{\theta\in[-\pi,\pi]} u(re^{i\theta})e^{-in\theta}"] \\
 & \begin{array}{c} l^2(\mathbb{Z},\mathbb{R}) \\ (\alpha_n)_{n\in\mathbb{Z}} \end{array} &
\end{tikzcd}
$$

commutes and each map is an unitary isomorphism of Hilbert spaces ?is the right morphism also unitary?

$$\frac{1}{i(e^{i\theta}-re^{it})}$$

$$\int_D |u|^2 < \infty$$

$$u(re^{i\theta}) = \int_{rz} \mathrm{d}u$$

$$u(re^{i\theta}) = \int_{ae^{i\theta}} X^\sharp$$

## when boundary value is $L^1$

not unique [1]

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Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$
    - Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
    - Ramified germs of smooth and holomorphic functions
    - Open subsets of $\mathbb{R}^n$
    - Open subsets of $\mathbb{R}^n$ equipped with the flat metric
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
    - Quasi-analytic smooth functions on $\mathbb{R}$
    - Star-shaped subsets of $\mathbb{R}^n$
    - ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$

1. multivariable calculus - Does the Poisson kernel give a unique harmonic function with given boundary data? - Mathematics Stack Exchange ↩