Info

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Limit sets of discrete subgroups

> Let
>
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a discrete subgroup.

Definition.

$$\Lambda(G) := \overline{G\{x\}} \cap \partial\mathbb{R}\boldsymbol{H}^n$$

Proposition:

> Let
>
> $$G \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a closed subgroup.

- Fixed points are in the limit set

$$\mathrm{Fix}(G) \cap \partial\mathbb{R}\boldsymbol{H}^n \subseteq \Lambda(G)$$

- The limit set is closed subset of $\partial\mathbb{R}\boldsymbol{H}^n$.
- The limit set is G-invariant

$$G(\Lambda(G)) = G$$

for $\Lambda(\Gamma) \neq \partial\mathbb{R}\boldsymbol{H}^n$

Theorem 12.2.5. *Let   be a nonelementary discrete subgroup of* $\mathrm{M}(B^n)$. *Then the limit set* $L(\ \ )$ *is perfect and is therefore uncountable.*

Theorem 12.2.11. *If P is a convex fundamental polyhedron for a discrete subgroup   of* $\mathrm{M}(B^n)$, *then* $O(\ \ ) = \cup\{g(\overline{P} \cap O(\ \ )) : g \in \ \ \}$.

> Let
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> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a discrete subgroup.

Then $\Lambda(\Gamma) \neq \partial\mathbb{R}\boldsymbol{H}^n$

$$\Longleftrightarrow$$

every convex fundamental polyhedron for Γ contains a half-space of $\mathbb{R}\boldsymbol{H}^n$

$$\Longleftrightarrow$$

Γ has a convex fundamental polyhedron that contains a half-space of $\mathbb{R}\boldsymbol{H}^n$.

As a corollary we have

- Γ is of finite covolume $\implies$ convex fundamental polyhedron does not contain a half-space $\implies$ $\Lambda(\Gamma) = \mathbb{R}\boldsymbol{H}^n$

limit set of a non-elementary group agrees with the same for an infinite normal subgroup

> Let
>
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a discrete subgroup.

If Γ is non-elementary and $H \trianglelefteq \Gamma$ is an infinite normal subgroup then

$$\Lambda(H) = \Lambda(\Gamma)$$

As a consequence

- For

$$L \underbrace{\leq}_{\text{finite index}} \underbrace{H}_{\text{infinite}} \underbrace{\trianglelefteq}_{\text{infinite index}} \underbrace{\Gamma}_{\text{finite covolume}}$$

  we have L is of infinite covolume, but $\Lambda(L) = \Lambda(H) = \Lambda(\Gamma) = \partial\mathbb{R}\boldsymbol{H}^n$.
  - In particular, convex fundamental domain of L does not contain a half-plane but has infinite volume.

limit set recognizes the smallest $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^m)$ that Γ is a subgroup of

> Let
>
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a discrete subgroup.

Let $1 \leq m \leq n$ be the smallest integer such that $\Lambda(\Gamma)$ is contained in an $m-1$-sphere of $\partial\mathbb{R}\boldsymbol{H}^n \cong_{\mathsf{Top}} S^n$. By conjugating Γ we may assume

$$\Lambda(\Gamma) \subseteq S^{m-1} = \partial \mathbb{R}\boldsymbol{H}^m$$

Then

$$\begin{aligned} &\Gamma \curvearrowright \operatorname{convex}(\Lambda(\Gamma)) \\ \implies &\Gamma \curvearrowright \operatorname{convex}(\Lambda(\Gamma)) \cap \mathbb{R}\boldsymbol{H}^n \subseteq \mathbb{R}\boldsymbol{H}^m \end{aligned}$$

But if $\Gamma(\mathbb{R}\boldsymbol{H}^m) \nsubseteq \mathbb{R}\boldsymbol{H}^m$ then $\operatorname{convex}(\Lambda(\Gamma)) \cap \mathbb{R}\boldsymbol{H}^n$ lies in a smaller $\mathbb{R}\boldsymbol{H}^k \subset \mathbb{R}\boldsymbol{H}^m$ but this means $\Lambda(\Gamma)$ is in a $k-1$-sphere. Thus

$$\implies \Gamma \curvearrowright \mathbb{R}\boldsymbol{H}^m$$

Thus

$$\Gamma \hookrightarrow \operatorname{Isom}(\mathbb{R}\boldsymbol{H}^m) \hookrightarrow \operatorname{Isom}(\mathbb{R}\boldsymbol{H}^n)$$

Current note has 1 direct children and 1 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Limit sets of discrete subgroups
        - Conical limit set

And it has 11 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Convex cocompact discrete subgroups
      - Combination of discrete subgroups of $O^+(1,n)(\mathbb{R})$
      - Exponential sum of a discrete subgroup of $\operatorname{Isom}(\mathbb{R}\boldsymbol{H}^n)$
      - $\operatorname{RepDF}(F_g, O^+(1,n)(\mathbb{R}))$
      - Measure homology on a hyperbolic manifold
      - Discrete subgroups with limit set a discontinuum
      - Limit sets of discrete subgroups
      - Discrete subgroups with limit set a $k-1$-sphere
      - Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$
      - Rigidity of $\mathbb{R}$-hyperbolic manifolds
      - $\operatorname{RepDF}(\pi_1(T_g^2, \bullet), O^+(1,n)(\mathbb{R}))$