Info

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Complement of knots in $\mathbb{R}^3$ or S^3

We want to study

Definition. Let K be a knot in S^3 then the open set

$$S^3 \setminus K$$

is called the **knot complement**.

Its a 3-manifold.

homology

Let T be the tubular nbd of $K \subseteq S^3$ which deformation retracts to S^1 thus

$$H_0(T) \cong \mathbb{Z},\ H_1(S^1) \cong \mathbb{Z},\ H_{k\geq 2}(S^1) = 0$$

Let $X := S^3 \setminus \mathrm{int}(T)$ then

$$X \sim S^3 \setminus K,\ X \cap T \cong S^1 \times S^1$$

and $S^3 = X \cup T$.

$$H_k(S^3 \setminus K) \cong \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & k \geq 2 \end{cases}$$

so this cannot be an invariant :(

linking number

stem.knots.R3.inv linking

Linking number invariant of links

Definition. Linking number invariant of links

In S^3 with two knots K_1, K_2 (linked) we have

$$[K_2] \in \pi_1(S^3 \setminus K_1)_{\mathsf{Ab}} \cong \mathbb{Z}$$

hence $[K_2]$ maps to an integer. The linking number is that integer.

On the infinite cyclic cover $\tilde{X} \to S^2 \setminus K_1$ let $\tilde{K}_2$ denote the lift of the knot K_2 onto $\tilde{X}$ and let τ denote the generator of the deck group then

$$\tau^n(x_0) = K_2(1)$$

where n is the linking number of K_1 and K_2.

Let Σ_1 denote the Seifet surface of K_1 such that K_2 intersects transversally at every point in $K_2 \cap \Sigma_1$ we assign $+1, -1$ depending on whether it passes from N_+ or N_-. Then the sum of these numbers is the linking number.

$$\coprod Y_i$$

There is an *unique* non-degenerate bilinear form

$$H_1(S^3 \setminus \Sigma) \times H_1(\Sigma) \to \mathbb{Z}$$

that produces

$$\beta([K_1], [K_2]) = \mathrm{link}(K_1, K_2)$$

Definition. The **Sifet matrix**

$$\begin{aligned} \alpha : H_1(\Sigma) \times H_1(\Sigma) &\to \mathbb{Z} \\ ([x], [y]) &\mapsto \beta([x^+], [y]) \end{aligned}$$

where x^- denotes x pushed outward into $S^3 \setminus \Sigma$

Start with the basis $[f_i], [e_j]$ for $H_1(S^3 \setminus \Sigma)$ and $H_1(\Sigma)$ such that

$$\beta([e_i], [f_j]) = \delta_{ij}$$

then

$$[f_i^-] = \sum \lambda_{ij} [e_j]$$

and by definition

$$\alpha([f_i], [f_j]) = \beta([f_i^-], [f_j]) = \lambda_{ij}$$

Therefore

$$[\alpha]_{([f_i^-])} = \lambda_{ij} \leftrightarrow [f_i^-] = \sum \lambda_{ij} [e_j]$$

which is the *matrix for the Seifert form* called the **Seifert matrix**.

Infinite cyclic cover of $S^3 \setminus K$

- We have the knot complement $X := S^3 \setminus K$.

- Let there exist a bicollar neighbourhood of $\Sigma \setminus K$ in S^3

  $$N : (\Sigma \setminus K) \times (-1, 1) \to S^3$$

  and define $N^+ := N((\Sigma \setminus K) \times (-1, 0))$ and $N^- := N((\Sigma \setminus K) \times (0, 1))$ with $\phi : N^- \to N^+$

$$Y$$

$$N^- \qquad N^+$$

  - Thus we have $X = \frac{Y}{\phi}$
- Let $Y_i := S^3 \setminus \Sigma$ and $h_i : Y_i \to S^3 \setminus \Sigma$ be the identity homeomorphism.
  - $$\tilde{X} := \frac{\coprod_{i \in \mathbb{Z}} Y_i}{}$$

  $$Y_0$$

  $$N_0^- \qquad N_0^+$$

  -

homology

$$Y' := \coprod_{i \in \mathbb{Z}} \overline{Y}_{2i}$$
$$Y'' := \bigcup_{i \in \mathbb{Z}} \overline{Y}_{2i+1}$$

then

$$Y' \cap Y'' = \coprod \Sigma_i$$

Now we have a pair, Mayer–Vietoris it!

$$\begin{array}{ccccccc} 0 \to & C_k(Y' \cap Y'') & \to & C_k(Y') \oplus C_k(Y) & \to & C_k(\tilde{X}) & \to 0 \\ & z & \mapsto & (-z, z) & & & \\ & & & (x, y) & \mapsto & x + y & \end{array}$$

this is short exact that commutes with the action of the $\mathbb{Z}\mathrm{Deck}(\tilde{X})$ which is isomorphic to $\mathbb{Z}\langle t \rangle \cong \mathbb{Z}[t, t^{-1}]$, hence *short exact as a* $\mathbb{Z}[t, t^{-1}]$*-module.*

This gives a exact sequence of $\mathbb{Z}[t, t^{-1}]$-modules

$$\begin{array}{ccccccc} \to & H_3(Y' \cap Y'') & \to & H_3(Y') \oplus H_3(Y) & \to & H_3(\tilde{X}) \to \\ \to & H_2(Y' \cap Y'') & \to & H_2(Y') \oplus H_2(Y) & \to & H_2(\tilde{X}) \to \\ \to & \underbrace{H_1(Y' \cap Y'')}_{\mathbb{Z}[t,t^{-1}] \otimes \mathbb{Z}^{2g+n-1}} & \xrightarrow{\alpha_*} & H_1(Y') \oplus H_1(Y) & \xrightarrow{\beta_*} & H_1(\tilde{X}) \to \\ \to & H_0(Y' \cap Y'') & \to & H_0(Y') \oplus H_0(Y) & \to & H_0(\tilde{X}) \end{array}$$

notice

$$\alpha_*([f_i]) = \sum A_{ij}[e_j] + \sum t A_{ji}[e_j]$$

- β_* is surjection, so we have

  $$H_1(\tilde{X}) \cong_{\mathbb{Z}[t, t^{-1}]} \frac{H_1}{H_1}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - Knots
    - Knots in $\mathbb{R}^3$
      - Complement of knots in $\mathbb{R}^3$ or S^3

And it has 5 siblings.

- stem
  - Knots
    - Knots in $\mathbb{R}^3$
      - Left trefoil knot
      - Complement of knots in $\mathbb{R}^3$ or S^3
      - Genus of a knot
      - Fundamental group of the complement of the knot
      - Linking number invariant of links