Info

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Complex ODEs of order 2 in $\mathbb{C}$

If $z_0 \in U$ such that

$$\begin{aligned} &a_0 w'' + a_1 w' + a_2 w = 0 \\ &a_i : U \to \mathbb{C} \text{ holomorphic} \\ &a_0(z_0) = 0 \end{aligned}$$

then there exists a *local* solution (f, Ω) with $z_0 \in \Omega$ for each choice of

$$f(z_0) = b_0,\ f'(z_0) = b_1$$

and the germ (f, z_0) is uniquely determined.

Write the equation in the form

$$w'' = p(z)w' + q(z)w$$

where p, q are analytic in a nbd of $z_0 = 0$. Let

$$\begin{aligned} p(z) &= p_0 + p_1 z + \cdots + p_n z^n + \ldots \\ q(z) &= q_0 + q_1 z + \cdots + q_n z^n + \ldots \end{aligned}$$

and now chose $b_0, b_1 \in \mathbb{C}$ and define

$$\begin{aligned} &f(z) = b_0 + b_1 z + \cdots + b_n z^n + \ldots \\ &2b_2 = b_1 p_0 + b_0 q_0 \\ &6b_3 = 2b_3 p_0 + b_1 p_1 + b_1 q_0 + b_0 q_1 \\ &\ldots \\ &n(n-1)b_n = \textstyle\sum_{i=1}^{n} (n-i) b_{n-i} p_{i-1} \end{aligned}$$

which proves uniqueness. Now we must show has a positive radius of convergence. Since p, q have a positive radius of convergence, for any $M_0, r_0 > 0$ we have

$$\begin{aligned} |p_n| &\le M_0 r_0^{-n} \\ |q_n| &\le M r_0^{-n} \end{aligned}$$

- Choose $M, r > 0$ such that

$$\begin{aligned} &|b_0| \le M \\ &|b_1| \le M r^{-1} \\ &r < r_0 \\ &M_0 \left(\tfrac{r}{2} + r^2\right) \le 1 \end{aligned}$$

- Let $|b_i| \le M r^{-i}$ for all $i < n$.
  - Then

$$\begin{aligned} n(n-1)|b_n| &\le M M_0 (1 + 2 + \cdots + (n-1) r^{1-n} + (n-1) r^{2-n}) \\ &= M M_0 \left(\frac{n(n-1)}{2} r + (n-1) r^2\right) U r^{-n} \\ |b_n| &\le M M_0 \left(\frac{r}{2} + \frac{r^2}{n}\right) r^{-n} \\ &\le M M_0 \left(\frac{r}{2} + r^2\right) r^{-n} \\ &\le M r^{-n} \end{aligned}$$

The space of solutions is a 2 complex-dimensional vector space

$$\begin{aligned} \{w : a_0 w'' + a_1 w' + a_2 w = 0\} &\to \mathbb{C}^2 \\ w &\mapsto (w(z), w'(z)) \end{aligned}$$

is an isomorphism.

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regular singular points

solutions at ∞

By the transformation

$$z \mapsto Z := \frac{1}{z}$$

the equation

$$w'' = p(z)w' + q(z)w$$

becomes

$$\frac{\partial^2 w}{\partial Z^2} = -\left(\frac{2}{Z} + \frac{p\left(\frac{1}{Z}\right)}{Z^2}\right)\frac{\partial w}{\partial Z} + \frac{q\left(\frac{1}{Z}\right)}{Z^4} w$$

- ∞ is an ordinary point $\iff$

$$-(2z + z^2 p(z)),\ z^4 q(z)$$

  has removable singularities at $z = \infty$ or $Z = 0$

$$\iff$$

  $p(z), q(z)$ has a simple and a double pole at $z = \infty$ or $Z = 0$

	$p(z)$	$q(z)$	**solution(s)**
		$q(z)$ must have at least 4 poles, unless it vanishes identically	
	$p(z)$ can have as few as 1 pole		
	the 1, simple pole is placed at $z = 0$ then $$p(z) = -2/z$$	$q = 0$	$$w'' = \left(-\frac{2}{z}\right) w' + \cancel{q(z}$$
	$$p(z) = \frac{A}{z}$$	$$q(z) = \frac{B}{z^2}$$	

Current note has 0 direct children and 0 total descendants.

- stem
    - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
        - Complex ODEs in $\mathbb{C}$
            - Complex ODEs of order 2 in $\mathbb{C}$

And it has 1 siblings.

- stem
    - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
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