Info

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Locally compact Hausdorff spaces

Definition. A Hausdorff space is **locally compact** if every point has a open neighborhood whose closure is compact.

> **2.4 Theorem** *Suppose K is compact and F is closed, in a topological space X. If $F \subset K$, then F is compact.*
>
> PROOF If $\{V_\alpha\}$ is an open cover of F and $W = F^c$, then $W \cup \bigcup_\alpha V_\alpha$ covers X; hence there is a finite collection $\{V_{\alpha_i}\}$ such that
>
> $$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$
>
> Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////
>
> **Corollary** *If $A \subset B$ and if B has compact closure, so does A.*
>
> **2.5 Theorem** *Suppose X is a Hausdorff space, $K \subset X$, K is compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \varnothing$.*
>
> PROOF If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q, such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \ldots, q_n \in K$ such that
>
> $$K \subset V_{q_1} \cup \cdots \cup V_{q_n}.$$
>
> Our requirements are then satisfied by the sets
>
> $$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad \text{and} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}. \qquad ////$$
>
> **Corollaries**
>
> (*a*) *Compact subsets of Hausdorff spaces are closed.*
> (*b*) *If F is closed and K is compact in a Hausdorff space, then $F \cap K$ is compact.*
>
> Corollary (*b*) follows from (*a*) and Theorem 2.4.

$[0,\infty]$-measures and positive functions on $\mathcal{C}_c$

A locally finite positive Borel measure μ on X gives a functional

$$\begin{aligned}\Lambda : \mathcal{C}_c(X) &\to k\\ \Lambda(f) &:= \int_{(X,\mu)} f\end{aligned}$$

Thus functional is *positive* in the sense that if $f(X) \subset (0,1)$ then $(\Lambda f)(X) \subset (0,1)$

> $$\mathrm{locFinMes}(\mathfrak{B}_X, [0,\infty]) \to \mathcal{C}_c(X)^{*>0}$$
>
> .

[1]

The converse of this statement is true for a much *regular* class of measures!

(Riesz representation of positive functionals on $\mathcal{C}_c$) Let X be locally compact Hausdorff space and Λ be a positive linear functional on $\mathcal{C}_c(X)$. Then there exists a σ-algebra $\mathfrak{M}_\Lambda$ in X which contains all Borel sets $\mathfrak{B}_X$ in X and there exists a **unique positive measure**

$$\mu : \mathfrak{M}_\Lambda \to [0,\infty]$$

such that $\Lambda(f) = \int_X f \mathrm{d}\mu$ for every $f \in \mathcal{C}_c(X)$, we have

$$\begin{aligned}\text{compact } K \subseteq X &\implies \mu(K) < \infty\\ E \in \mathfrak{M}_\Lambda &\implies \mu(E) = \inf\{\mu(V) | E \subseteq V, V \text{ open } \subseteq X\}\\ \begin{cases} E \text{ open}\\ \text{or}\\ E \in \mathfrak{M}_\Lambda, \mu(E) < \infty\end{cases} &\implies \mu(E) = \sup\{\mu(K) | K \subseteq E, K \text{ cpt } \subseteq X\}\end{aligned}$$

and if $E \in \mathfrak{M}_\Lambda, A \subseteq E, \mu(E) = 0$ then $A \in \mathfrak{M}_\Lambda$.

Thus

$$\mathcal{C}_c(X,\mathbb{R})^{*>0} \simeq_{\mathsf{Set}} \mathrm{RieszMes}(\mathfrak{B}_X, [0,\infty])$$

Borel $\mathbb{C}$-measures

(Riesz representation for Borel $\mathbb{C}$-measures) Let X be a locally compact Hausdorff space. Then every bounded linear functional

$$\varphi : \mathcal{C}_0(X) \to \mathbb{C}$$

is represented by a unique regular complex Borel measure μ, that is,

$$\forall f \in \mathcal{C}_0(X), \varphi(f) = \int_X f \mathrm{d}\mu$$

Moreover, the norm of φ is the total variation of μ

$$\|\varphi\| = |\mu|(X)$$

$$\mathcal{C}_0(X,\mathbb{C})^\star \simeq_{\mathsf{Ban}_\mathbb{C}} (\mathrm{regMes}(\mathfrak{B}_X,\mathbb{C}), |\,|(X))$$

Borel $[0,1]$-measures

We have continuous maps

$$\mathrm{ev}_f : \mathcal{C}_0(X, \mathbb{C})^\bigstar \to \mathbb{C}$$

so the subset of positive functionals

$$\mathcal{C}_0(X, \mathbb{C})^{\bigstar > 0} := \bigcap_{f \geq 0} \mathrm{ev}_f^{-1}[0, \infty]$$

is an intersection of closed sets, so it is closed.

$$\mathcal{C}_0(X, \mathbb{C})^{\bigstar > 0} \cong \mathrm{regMeas}(\mathfrak{B}_X, [0, \infty))$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Locally compact Hausdorff spaces

And it has 29 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
    - Discrete subgroup acting on coset space
    - Basis and subbasis for a topology
    - Principal G-bundles over topological spaces

$$G \curvearrowright P \to X$$

    - Vector bundles over topological spaces
    - Closure of a subset of topological space
    - Locally compact and 2-countable Hausdorff topological spaces are paracompact with countable refinement
    - Covering dimension of topological spaces
    - $\mathrm{EndTop}(X, Y)$
    - Eilenberg–Steenrod functors

$$(H_n : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Ab}, \partial_n)$$

    - Local fields
    - Local fields of characteristic 0
    - Topological groups
    - Continuous group action on a topological space
    - Intersection of open, dense subsets of a topological space
    - Local topological spaces
    - Covering space of a topological space
    - Borel measures on a topological space
    - Convergence for nets in a topological space
    - Loops and a open cover of a topological space
    - Paths in a topological space
    - Product of topological spaces
    - Ringed topological spaces
    - Sheaf on open sets of a topological space
    - Closed subsets of a topological space
    - Discrete subsets of topological spaces
    - Types of topological spaces
    - Locally compact Hausdorff spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$

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1. math.uchicago.edu/~may/REU2023/REUPapers/Espejo.pdf#page=9.37 ↩