ⓘ Info

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Geodesics in a metric space

Definition. Arcs, rays and lines

A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called	if
arc/segment joining $p, q \in M$	$I = [a, b]$ and $\gamma(0) = p, \gamma(1) = q$
ray originating at $p \in M$	$I = [0, \infty)$ and $\gamma(0) = p$
line	$I = \mathbb{R}$

Definition. Geodesics in a metric space

Let (M, d) be a metric space. A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called	if
minimal geodesic	$$\forall t_1, t_2 \in I, d(\gamma(t_1), \gamma(t_2)) = \lvert t_1 - t_2 \rvert$$
unit speed (local) geodesic (locally (in time) minimal geodesic)	$\forall t \in I$ there exists a $\epsilon > 0$ such that $$d(\gamma(t_1), \gamma(t_2)) = \lvert t_1 - t_2 \rvert$$ for all $t_1, t_2 \in (t - \epsilon, t + \epsilon)$.

Therefore, we have

- *minimal geodesic segments* which are isometric embeddings $[0, L] \to M$
- *minimal geodesic rays* which are isometric embeddings $[0, \infty) \to M$
- *minimal geodesic lines* which are isometric embeddings $\mathbb{R} \to M$

Definition. Geodesic metric spaces

A metric space (M, d) is

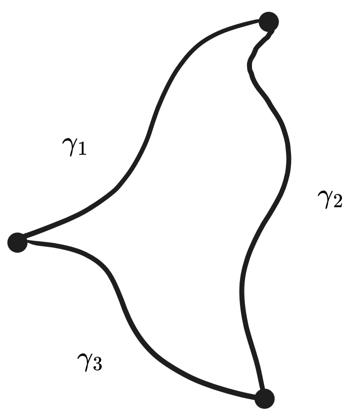
- a **geodesic metric space** or is **geodesically connected** if for every pair of points $p, q \in M$ there exists a *minimal geodesic* $\gamma : [a, b] \to M$ that joins them $\gamma(a) = p, \gamma(b) = q$
- **uniquely geodesic** if there is exactly *one* minimal geodesic joining x to y for each $x, y \in M$, denoted by $\overline{xy}$
- **geodesically complete** if every *minimal geodesic arc* $\gamma : [a, b] \to M$ can be extended to a *minimal geodesic line* $\tilde{\gamma} : \mathbb{R} \to M$.

Definition. Geodesic triangles

Let (X, d) be a metric space. A **geodesic triangle** is a triple

$$\gamma_1, \gamma_2, \gamma_3 : [0, L_1], [0, L_2], [0, L_3] \to X$$

of *geodesic segments*



such that

$$\gamma_1(L_1) = \gamma_2(0)$$
$$\gamma_2(L_2) = \gamma_3(0)$$
$$\gamma_3(L_3) = \gamma_1(0)$$