Info

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S^3 as two solid tori glued at their boundaries

The 3-sphere S^3 decomposes into two solid tori $S^1 \times D^2$ and $D^2 \times S^1$.

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2$$

covering S^3 with two solid tori

- We take the 3-sphere

  $$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^2$$

  and consider the closed set

  $$\begin{aligned} N_1 &:= \left\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 \mid |z_1|^2 \geq |z_2|^2\right\} \\ &= \left\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 \mid |z_1|^2 \geq \frac{1}{2}\right\} \\ &= \left\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 \mid |z_2|^2 \leq \frac{1}{2}\right\} \end{aligned}$$

- And note the closure of its complement

  $$N_2 := \left\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 \mid |z_1|^2 \leq |z_2|^2\right\}$$

- Of course N_1 and N_2 **cover** S^3

  $$\begin{aligned} S^2 &= \{|z_1|^2 \geq |z_2|^2\} \cup \{|z_1|^2 \leq |z_2|^2\} \\ &= N_1 \cup N_2 \end{aligned}$$

- Their intersection

  $$N_1 \cap N_2 = \left\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 \mid |z_1| = |z_2| \left(= \frac{1}{\sqrt{2}}\right)\right\}$$

  is homeomorphic to the 2-torus as

  $$N_1 \cap N_2 = \frac{1}{\sqrt{2}} S^1 \times S^1 \subseteq \mathbb{C}^2$$

- We have the bijection [1]

  $$\begin{aligned} \pi_1 \times P_2 : N_2 &\to \frac{1}{\sqrt{2}} D^2 \times S^1 \\ (z_1, z_2) &\mapsto \left(z_1, \frac{z_2}{|z_2|}\right) \\ (\alpha, \sqrt{1 - |\alpha|^2}\beta) &\leftarrow (\alpha, \beta) \end{aligned}$$

  - As the projection π_1 onto first factor and projection onto the circle P_2 are smooth maps, their product is smooth.
  - The inverse map is $\pi_1 \times f\pi_2$ scaling where

    $$\begin{aligned} f : \tfrac{1}{\sqrt{2}} D^2 \times S^1 \quad &\to 1 - \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] = \left[\tfrac{1}{2}, 1 + \tfrac{1}{2}\right] \quad &&\to \mathbb{R} \\ (\alpha, \beta) \quad &\mapsto 1 - |\alpha|^2 &&\mapsto \sqrt{1 - |\alpha|^2} \end{aligned}$$

    is a composition of smooth functions, this is smooth.
  - Hence, the map $N_2 \to \frac{1}{\sqrt{2}} D^2 \times S^1$ is a diffeomorphism.
- Consider the *reflection*

  $$\begin{aligned} R : \mathbb{C}^2 &\to \mathbb{C}^2 \\ (z_1, z_2) &\mapsto (z_2, z_1) \end{aligned}$$

  - This is a diffeomorphism as it is a linear bijective map whose inverse is itself.
  - Under this map, the inequalities $|z_1|^2 \geq |z_2|^2$ flip so it restricts to a map

    $$R : N_1 \leftrightarrow N_2$$

    which is a diffeomorphism.

Current note has 0 direct children and 0 total descendants.

And it has 2 siblings.

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1. https://mathoverflow.net/a/228701 ↩