Info

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S^2 and $\mathbb{C}\boldsymbol{P}^1$

Definition. S^2

The 2-sphere S^2 is the topological space

$$S^2 := \{(x,y,z) \in \mathbb{R}^3 \big| x^2+y^2+z^2=1\}$$

[1]

as a subset of $\mathbb{R}^3$

(Borsuk-Ulam theorem) Let $f : S^2 \to \mathbb{R}^2$ be a continuous function. Then there exists a $x \in S^2$ such that

$$f(x) = f(-x)$$

smooth manifold S^2

- S^2 is the *unique* simply connected connected smooth 2-manifold up to diffeomorphism apart from $\mathbb{R}^2$.
- S^2 is orientable.
- S^2 is the *unique* orientable genus 0 connected smooth 2-manifold.
- The vector bundle

$$\begin{array}{c} TS^2 \\ \downarrow \\ S^2 \end{array}$$

  is not trivial by

  (Cannot comb a hairy 2-sphere) Every continuous tangent vector field on S^2 has a zero.
- Therefore, S^2 does not admit a Lie group structure.
- By

  Let a **compact Lie group** G act smoothly, faithfully on a *connected* smooth manifold M. Then

$$\dim G \le \frac{1}{2} \dim M(\dim M + 1)$$

  this 3 is the largest dimension of a *compact* Lie group that can act on S^2, which is attained by the $SO(3,\mathbb{R}) \curvearrowright S^2$.

Riemannian manifold $(S^2, \langle,\rangle_{S^2})$

Definition. Round metric on S^2

The round metric $\langle,\rangle_{S^2}$ on S^2 is the pullback of the flat metric $\langle,\rangle$ on $\mathbb{R}^3$ through the inclusion $S^2 \hookrightarrow \mathbb{R}^3$.

Riemann surface $\mathbb{C}\boldsymbol{P}^1$

The complex 2-manifold $\mathbb{C}\boldsymbol{P}^1$ is the unique one which is homeomorphic/diffeomorphic to S^2.

Definition. Mobius transformations on $\mathbb{C}P^1$

The action $SL(2)(\mathbb{C})$ on $\mathbb{C}^2$ descends to the action of $PSL(2)(\mathbb{C})$ acts on $\mathbb{C}P^1$ by **Mobius transformations**

$$PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\text{Man}} S^2 \cong \mathbb{C} \cup \{\infty\}$$

$$z \xmapsto{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \frac{az+b}{cz+d}$$

conformal manifold S^2

$$O^+(1,3)(\mathbb{R}) \cong \text{Moeb}(S^2) \curvearrowright S^2$$

as the conformal boundary of $\mathbb{R}\boldsymbol{H}^3$

symplectic manifold (S^2, λ_{S^2})

The symplectic manifold (S^2, λ_{S^2}) is a non-exact homogeneous symplectic manifold.

- It is the coadjoint orbit of $SO(3)(\mathbb{R})$.

Kahler manifold $(\mathbb{C}\boldsymbol{P}^2, \langle,\rangle_{S^2} + i\lambda_{S^2})$

as a real algebraic curve

as universal covering of $\mathbb{R}\boldsymbol{P}^2$

as the quotient of $U(1) \curvearrowright S^3 \subseteq \mathbb{C}^3$

Current note has 10 direct children and 10 total descendants.

- stretchy
  - S^2 and $\mathbb{C}\boldsymbol{P}^1$
    - Almost complex structure on S^2 induced from spherical metric and its area form
    - Borsuk-Ulam theorem
    - Cylindrical coordinates on S^2
    - Holomorphic vector bundles on $\mathbb{C}\boldsymbol{P}^1$
    - Round metric on S^2
    - The area form on round S^2
    - Stereographic coordinates on the sphere S^2
    - TS^2
    - Vector fields on S^2
    - Hamiltonian flows on the sphere S^2

And it has 62 siblings.

- stretchy
  - Trefoil 3_1 knot
  - The only compact connected Lie group that can act faithfully on a torus is itself
  - Hecke algebras
  - $\overline{B^n}$
  - $\mathrm{BS}(1,2)$
  - $(\mathbb{C},+,\times)$
  - $D^* := D \setminus \{0\}$
  - Cantor set
  - Configuration space of N rigid points in $\mathbb{R}^3$
  - Free groups
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
  - H_2^3
  - $[a,b]$
  - Figure 8 knot
  - Lamplighter group on $\mathbb{Z}_2$
  - $\beta\mathbb{N}$
  - $\#_2\mathbb{R}P^2$
  - $\prod_{i\in\mathbb{N}}\{1,\ldots,n_i\}$
  - $\mathbb{R}P^n$
  - $(\mathbb{Q},+,\times,<)$
  - $\mathbb{Q}_{\hat{p}}$
  - $\overline{\mathbb{Q}}$
  - $\mathbb{Q}_{\mathrm{constr}}$
  - $\mathbb{Q}(\xi_n)$
  - $\mathcal{O}_{\overline{\mathbb{Q}}}$
  - $(\mathbb{R},+,\times,<,\sup)$
  - Homeomorphism of $\mathbb{R}^2$ that sends $\mathbb{N}$ to a countable closed discrete subset
  - $\mathbb{R}^2 \times S^1$
  - $\mathbb{R}^n$
  - $\mathbb{R} \times S^1$
  - $\mathbb{R} \times S^2$
  - S^1
  - $S^1 \wedge S^1$
  - $S^1 \times S^n$
  - S^2 and $\mathbb{C}\boldsymbol{P}^1$
  - Mapping torus of the antipodal map of S^2
  - S^3
  - S^n
  - I-bundle
  - Symmetric group
  - $\mathbb{T}$
  - T^2
  - T_g^2
  - $T_{0,0,3}^2$
  - $\mathbb{R}^2 \setminus \{0\}$
  - Three $T_{1,0,1}^2$-glued at their boundaries
  - $T_{1,1,0}^2$
  - T_2^2
  - $T_g^2 \times [0,1]$
  - $\#_{\mathbb{N}}T^2$
  - Thompson's group
  - T^n
  - 3-valent tree
  - 4-valent tree
  - $[0,1]^{\mathbb{N}}$ with product topology
  - Whitehead manifolds
  - $\mathbb{Z}$
  - $\mathbb{Z}[\sqrt{m}]$
  - $\mathbb{Z}[X]$
  - $\mathbb{Z}^{\mathbb{N}}$
  - $\mathbb{Z}[i]$
  - $\mathbb{Z}[i\sqrt{5}]$

1. https://chessapig.github.io/diary/21 ↩