Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on October 24, 2023 4:23:12 PM,
was last modified on August 29, 2026 1:00:50 PM, and
was exported on September 7, 2026 3:04:06 PM.

Loops in S^1 up to homotopy

lifting paths

- Let

$$q:[0,1]\to \mathbb{S}^1,\ q(0)=(1,0)$$

be a path starting at $(1,0)$. Then apply the

> For a
>
> > **Definition.** Covering space of a topological space
> >
> > A continuous map between topological spaces
> >
> > $$\pi: Y\to X$$
> >
> > is a **covering space map** if π is *surjective* and for any $x\in X$, $\exists U_x(\text{open})\subseteq X$ of x such that
> >
> > $$\pi^{-1}U_x=\bigsqcup_{a\in\pi^{-1}(x)} Y_a$$
> >
> > where $\pi|_{V_a}: V_a\to U_x$ is a homeomorphism for each $a\in\pi^{-1}(x)$.
>
> and continuous map
>
> $$h: Y\times[0,1]\to X$$
>
> then for *any chosen lift $\tilde{f}$ of the initial* $h(-,0):=f_0: Y\to X$
>
> $$\tilde{f}: Y\to\tilde{X}, \pi\circ\tilde{f}=f$$
>
> we have a **unique lift of** h
>
> $$\exists!\,\tilde{h}: Y\times[0,1]\to\tilde{X},\ \pi\circ\tilde{h}=h$$
>
> with the initial $\tilde{h}(-,0)=\tilde{f}$.

for

$$Y:=\{\bullet\}$$

the covering space map

$$\begin{aligned}\pi:\mathbb{R}&\to\mathbb{S}^1\\ t&\mapsto(\cos 2\pi t,\sin 2\pi t)\end{aligned}$$

for the initial lift

$$0:\{\bullet\}\to\mathbb{R}, \pi(0)=(1,0)$$

we must have an unique

$$\tilde{q}:[0,1]\to\mathbb{R},\ \tilde{q}(0)=0,\ \pi\circ\tilde{q}=q$$

  - Hence, we have unique lift $\tilde{q}$ of paths q starting at 0 given that $\tilde{q}(0)=0$.

lifting homotopies

- Let $p,q:[0,1]\to\mathbb{S}^1$ starting at $(1,0)$ lifts to $\tilde{q},\tilde{p}:[0,1]\to\mathbb{R}$ such that $\tilde{q}(0)=0=\tilde{p}(p)$. Let

$$\tilde{p}(1)=m, \tilde{q}=n\in\mathbb{Z}$$

  - Then working with $(\tilde{q}+m)(t):=\tilde{q}(t)+m$, we observe

$$\pi(\tilde{q}(t)+m)=q(t),\ \tilde{q}(0)+m=\tilde{p}(1)$$

  - Thus

$$\tilde{p}\cdot(\tilde{q}+m)$$

  is well-defined and

$$\pi(\tilde{p}\cdot(\tilde{q}+m))=p\cdot q,\ (\tilde{p}\cdot(\tilde{q}+m))(0)$$

  which means it is the unique lift of $p\cdot q$.

    - Hence, the map

$$\begin{aligned}\pi_1(\mathbb{S}^1,(1,0))&\to(\mathbb{Z},+)\\ [p]&\mapsto\tilde{p}(1)\end{aligned}$$

    is a group isomorphism.

> $$\pi_1(\mathbb{S}^1,(1,0))\cong_{\mathsf{Grp}}(\mathbb{Z},+)$$

Current note has 0 direct children and 0 total descendants.

- stretchy
  - S^1
    - Loops in S^1 upto homotopy

And it has 4 siblings.

- stretchy
  - S^1

- The map $z \mapsto z^n$ on the circle
- Left translation $S^1 \curvearrowright S^1$
- Homeomorphisms $S^1 \to S^1$
- Loops in S^1 upto homotopy