Exponential of vector fields

Definition. Flow and exponential of a vector field on a smooth manifold

Flow of a vector field $X \in \mathrm{Vec}(M)$ is the *flow* on M, that is, the smooth map

$$(t, -) \mapsto (\exp tX)(-)$$

that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp tX\Big|_{t=0}(p) = X_p, \ \ \exp 0 = \mathsf{Id}_M$$

Then X is the **generator of the flow** and

$$\exp X : M \to M$$

is the **exponential of the vector field** X.

Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp tX = X_{\exp tX}$$

commutators

- BCH
- $$\exp \alpha[X, Y] = \exp[\sqrt{\alpha} X, \sqrt{\alpha} Y] = [\exp \sqrt{\alpha} X, \exp \sqrt{\alpha} Y]$$

complete vector fields

A **compactly** supported smooth vector field on a smooth manifold is **complete**.
