ⓘ Info

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Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$

The Fourier summation/inversion

$$\begin{aligned} l^1(\mathbb{Z}) &\to \mathcal{C}(S^1) \\ (x_k) &\mapsto \sum_{n\in\mathbb{Z}} x_n e^{2\pi i n x} \end{aligned}$$

- is **well-defined** because the trigonometric series uniformly converges

$$\left\| \sum_{n\in\mathbb{Z}} e^{inh} x_n \right\|_\infty \leq \sum_{n\in\mathbb{Z}} |x_n| = \| x \|_1$$

- is **continuous** again because of the above inequality

$$\left\| \sum_{n\in\mathbb{Z}} e^{inh} x_n \right\|_\infty \leq \| x \|_1$$

  and also has the operator norm ≤ 1.
- has **operator norm** $= 1$ as

$$(0, 0, \ldots, \underbrace{1}_{n=1}, 0, 0, 0 \ldots) \mapsto 1$$

  and both are of norm 1.

We call the image

Definition. The **Weiner algebra** on the circle

$$\check{l}^1(S^1) := \left\{ f \in \mathcal{C}(S^1) \mid \hat{f} \in l^1(\mathbb{Z}) \right\}$$

The Fourier transform and *Fourier summation* above are inverses

$$(l^1(\mathbb{Z}), \| \|_1) \cong_{\mathsf{TopVec}_{\mathbb{C}}} (\check{l}^1(S^1), \| \|_\infty)$$

but we may put push the norm on l^1, then

$$(l^1(\mathbb{Z}), \| \|_1) \cong_{\mathsf{Ban}_{\mathbb{C}}} (\check{l}^1(S^1), \| \|_{l^1})$$

$\mathcal{C}^2(S^1) \subset \check{l}^1(S^1)$

- Let $f \in \mathcal{C}^2(S^1)$ or more generally, $f \in \mathcal{C}^1[0,1]$ with $f' \in L^1[0,1]$ and $f(0) = f(1), f'(0) = f'(1)$.
  - From
    - Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
      - Then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} \hat{f}(n) &= \int_{x\in[0,1]} f(x) e^{-2\pi i n x} \\ &= \int_{x\in[0,1]} f(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{-2\pi i n} \frac{\mathrm{d}}{\mathrm{d}x} e^{-2\pi i n x} \right) \\ &= f(0) - f(1) - \frac{1}{-2\pi i n} \int_{x\in[0,1]} f'(x) e^{-2\pi i n x} \end{aligned}$$

      - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f'}(n) = (2\pi i) n \hat{f}(n)$$

    we know

$$\hat{f}(n) = \frac{\widehat{f''}(n)}{(2\pi i n)^2}$$

  - Further, by
    - For $f \in L^1[0,1]$, the Fourier coefficients of f

$$\hat{f}(n) := \int_{x\in[0,1]} f(x) e^{-2\pi i n x}$$

      is bounded

$$\begin{aligned} \left| \hat{f}(n) \right| &\leq \left| \int_{x\in[0,1]} f(x) e^{-2\pi i n x} \right| \\ &\leq \int_{x\in[0,1]} |f(x)| \\ &= \|f\|_1 \end{aligned}$$

      by its L^1-norm.

    we have for $n \in \mathbb{Z} \setminus \{0\}$

$$\left| \hat{f}(n) \right| \leq \frac{\|f''\|_1}{(2\pi i)^2} \frac{1}{n^2}$$

  - Thus

$$\sum_{n\in\mathbb{Z}} \left| \hat{f}(n) \right| \leq \left| \hat{f}(0) \right| + \frac{\|f''\|_1}{(2\pi i)^2} \underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}}_{2\frac{\pi^2}{6}}$$

- Thus Fourier summation of f converges *absolutely*.

$\mathcal{C}^1(S^1) \subset \check{l}^1(S^1)$

- Let $f \in \mathcal{C}[0,1]$ with $f' \in L^1[0,1]$ and

$$f(0) = f(1)$$

  - Consider

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| = \left|\hat{f}(0)\right| + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(|n| \left|\frac{\hat{f}(n)}{n}\right|\right)$$
$$\leq \left|\hat{f}(0)\right| + \sqrt{\sum_n \frac{1}{|n|^2}} \sqrt{\sum_n \left|n^2 \hat{f}(n)\right|^2}$$

  - From
    - Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
      - Then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f}(n) = \int_{x\in[0,1]} f(x) e^{-2\pi i n x}$$
$$= \int_{x\in[0,1]} f(x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{-2\pi i n}\frac{\mathrm{d}}{\mathrm{d}x} e^{-2\pi i n x}\right)$$
$$= f(0) - f(1) - \frac{1}{-2\pi i n}\int_{x\in[0,1]} f'(x) e^{-2\pi i n x}$$

      - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f'}(n) = (2\pi i) n \hat{f}(n)$$

    we know

$$\widehat{f'}(n) = (2\pi i n)^2 \hat{f}(n)$$

  - By
    - **(Parseval's and Riesz-Fischer theorems)** Let $f \in L^2[0,1]$. Then the partial Fourier summation converges to f almost everywhere and in L^2

$$S_N f \xrightarrow{N\to\infty} f \qquad \text{ae and } L^2$$

      and the Fourier transform is a $\mathbb{C}$-linear bijection

$$\widehat{\ }: L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$$

      which satisfies $\|f\|_2 = \left\|\hat{f}\right\|_2$ (is an isometry).

    we know

$$\|f'\|_2 = \left\|\widehat{f'}\right\|_2 = |2\pi i| \sqrt{\sum_{n\in\mathbb{Z}} \left|n \hat{f}(n)\right|^2}$$

  - Therefore,

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| \leq \left|\hat{f}(0)\right| + \sqrt{\sum_{n\neq 0} \frac{1}{|n|^2}} \sqrt{\sum_{n\in\mathbb{Z}} \left|n^2 \hat{f}(n)\right|^2}$$
$$\leq \left|\hat{f}(0)\right| + \sqrt{\frac{2\pi^2}{6}} \left\|\widehat{f'}\right\|_2$$

- In general,

$$\mathcal{C}^\alpha(S^1) \subset \check{l}^1(S^1)$$

  for $\alpha > \frac{1}{2}$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Fourier
      - Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$

And it has 10 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Fourier
      - Fourier transform on $L^2(-A,A) \leq L^2(\mathbb{R})$
      - $\widehat{\ }: L^2(0,\infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathbb{U}})$
      - Fourier transform on $\mathbb{R}^n$
      - Fourier transform on S^1, Fourier series on $[0,1]$
      - Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$
      - Fourier transform of distributions on S^1
      - $\widehat{\ }: L^1[0,1] \to \mathcal{C}_0(\mathbb{Z}, \mathbb{C})$
      - $\widehat{\ }: L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$
      - Fourier transform of measurable subsets
      - Unsharpness principles