Info

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Comparing a function and it derivative

Lipshitz bounds and bounds on derivatives

bounded derivative $\iff$ Lipshitz

Let

$$f : U(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$$

be differentiable.

- Then f being M-Lipshitz implies

$$\left|\frac{f(y) - f(y+h)}{h}\right| \leq M \implies |f'(y)| \leq M$$

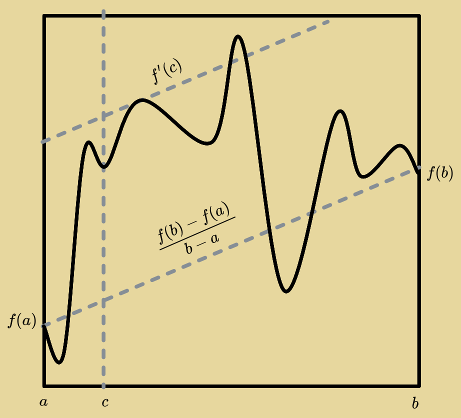
  so f has a bounded derivative on U.
- Let f have a bounded derivative on U, $|f'| \leq M$. Then for any $x, y \in U$, by

  (Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

  be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(c)$$



  we have

$$\exists c \in (x, y) : \left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)| \leq M$$

  Thus, f is M-**Lipschitz**.

$\mathcal{C}^1 \implies$ locally Lipschitz

Let

$$f : [a, b] \to \mathbb{R}$$

be differentiable and

$$f' : [a, b] \to \mathbb{R}$$

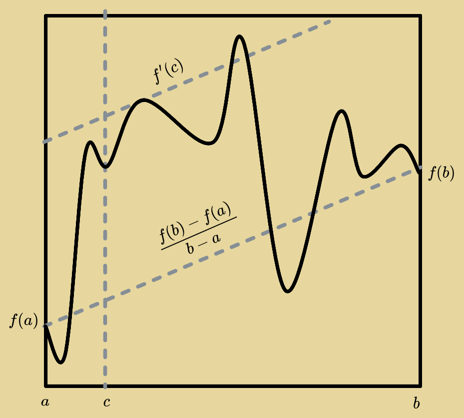
be continuous. Then f' is bounded, say $|f'| \leq M$. Then for any $x, y \in [a, b]$ by

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(c)$$



we have

$$\exists c \in (a, b) : \left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)| \leq M$$

Hence, f is **Lipshitz continuous** on $[a, b]$.

Comparing $\| \ \|_1$

- Let $f \in W^{1,1}(a, b)$. For any $\xi \in (a, b)$ we have

$$f(x) = f(\xi) + \int_\xi^x f'$$

- Then the derivation of f from $f(\xi)$ is bounded by

$$\begin{aligned}|f(x)-f(\xi)| &= \left|\int_\xi^x f'\right| \\ &= \left|\int_{[x,\xi]} f'\right| \\ &\leq \int_{(a,b)} |f'| \\ &= \|f'\|_1\end{aligned}$$

- This implies, in particular, we can bound f as well

$$\begin{aligned}\implies |f(x)| &\leq |f(\xi)| + \int_{(a,b)} |f'| \\ &= |f(\xi)| + \|f'\|_1\end{aligned}$$

- If $f(a)=0$ then we have

$$\|f\|_1 \leq \|f'\|_1 |b-a|$$

  - However, if $f(a)\neq 0$, then this inequality does not hold.

- However, this holds

$$\|f-f(a)\|_1 \leq \|f'\|_1 |b-a|$$

$$\begin{aligned}\mathcal{C}^1(0,1)\setminus\{0\} &\to [0,\infty) \\ f &\mapsto \frac{\|f\|_1}{\|f'\|_1}\end{aligned}$$

Comparing $\|\ \|_2$

- Let $f\in\mathcal{C}^1(a,b)$ such that $f'\in L^2(a,b)$.
  - Then by

    (Intermediate value theorem for continuous $[a,b]\to\mathbb{R}$) Let $f:[a,b]\to\mathbb{R}$ be continuous and suppose there are two points $\alpha<\beta$ in $[a,b]$ such that $f(\alpha)\neq f(\beta)$. Then $f(x)$ takes every value between $f(\alpha)$ and $f(\beta)$ for $x\in(\alpha,\beta)$. That is

    $$f(\alpha,\beta)\supseteq(f(\alpha),f(\beta))$$

    we have a $c\in[-r,r]$ such that

    $$f(c)=\frac{1}{2r}\int_{(-r,r)} f$$

  - Purring $\xi=c$ in
    - Then the derivation of f from $f(\xi)$ is bounded by

    $$\begin{aligned}|f(x)-f(\xi)| &= \left|\int_\xi^x f'\right| \\ &= \left|\int_{[x,\xi]} f'\right| \\ &\leq \int_{(a,b)} |f'| \\ &= \|f'\|_1\end{aligned}$$

    and using Cauchy-Swartz $\|f'\|_1\leq\left(\int_{(a,b)}1\right)\|f'\|_2$ we have

    $$|f(x)-f(\xi)|\leq|b-a|\|f'\|_2$$

    - Integrating

    $$\left\|f-\frac{1}{|b-a|}\int_{(a,b)}f\right\|_1\leq|b-a|^2\|f'\|_2$$

    which gives us nothing new.
    - Squaring and integrating, however, produces

    $$\left\|f-\frac{1}{|b-a|}\int_{(a,b)}f\right\|_2^2\leq|b-a|^2\|f'\|_2^2$$

We see

$f\in\mathcal{C}^1[a,b]$	then	and equality $\iff$
$\int_{[a,b]} f=0$ and $f(a)=fy(b)$	$\|f\|_2\leq\frac{\lvert b-a\rvert}{2\pi}\|f'\|_2$	$f(x)\equiv c\sin\frac{2\pi}{L}(x-a)$ for some $c,\alpha\in\mathbb{R}$
$f(a)=f(b)$	$\|f\|_2\leq\frac{\lvert b-a\rvert}{\pi}\|f'\|_2$	$f(x)\equiv c\sin\frac{\pi x}{L}$
$\int_{[a,b]} f=0$	$\|f\|_2\leq\frac{\lvert b-a\rvert}{\pi}\|f'\|_2$	$f(x)\equiv c\cos\frac{\pi x}{L}$

bounding $\|\ \|_p$ on domains with $\mathcal{C}^1$-boundary

(Poincare's inequality) Let $U\subset\mathbb{R}^n$ be a bounded, connected open subset with a $\mathcal{C}^1$-boundary ∂U. For $p\in[1,\infty]$, there exists $c(p,U)>0$ such that for every $u\in W^{1,p}(U)$ we have

$$\left\|u-\frac{1}{m(u)}\int_U u\right\|_p\leq c(p,U)\|\mathfrak{D}u\|_p$$

Current note has 0 direct children and 0 total descendants.

- stem
    - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
        - Differentiable functions
            - Comparing a function and it derivative

And it has 10 siblings.

- stem
    - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
        - Differentiable functions
            - Functions $(a, b) \to \mathbb{R}$ differentiable at a point
            - Comparing a function and it derivative
            - Absolutely continuous functions on $[a, b] \leftrightarrow \int_{[a,-]}(L^1[a, b])$
            - Distributional derivatives
            - Double derivative/Laplace operator on the circle
            - Fractional derivative
            - Infinite limit of derivatives
            - Space of continuous and continuously differentiable functions on $\mathbb{R}$
            - Derivative of maps $\mathbb{R}^n \to \mathbb{R}^m$
            - Zooming of a map $\mathbb{R}^n \to \mathbb{R}^m$