The boundary of symmetric spaces and discrete subgroups of isometry groups

PRJ501 presentation

Rupadarshi Ray
MS21165

December 1, 2025

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type	(Globally) symmetric spaces of non-compact type
	Let X be an irreducible symmetric space of non-compact type

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type		**(Globally) symmetric spaces of non-compact type**
Then $G := \mathrm{Isom}(X)$ is a non-compact (almost) simple Lie group	$\leftsquigarrow$	Let X be an irreducible symmetric space of non-compact type

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type		**(Globally) symmetric spaces of non-compact type**
Then $G := \mathrm{Isom}(X)$ is a non-compact (almost) simple Lie group	$\leftsquigarrow$	Let X be an irreducible symmetric space of non-compact type
Let G be an (almost) simple Lie group and $K \leqslant G$ be the maximal compact subgroup		

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type		(Globally) symmetric spaces of non-compact type
Then $G := \mathrm{Isom}(X)$ is a non-compact (almost) simple Lie group	$\leftsquigarrow$	Let X be an irreducible symmetric space of non-compact type
Let G be an (almost) simple Lie group and $K \leqslant G$ be the maximal compact subgroup	$\rightsquigarrow$	Then $X := G/K$ is an irreducible symmetric space of non-compact type

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type		(Globally) symmetric spaces of non-compact type
Then $G := \mathrm{Isom}(X)$ is a non-compact (almost) simple Lie group	$\leftsquigarrow$	Let X be an irreducible symmetric space of non-compact type
Let G be an (almost) simple Lie group and $K \leqslant G$ be the maximal compact subgroup	$\rightsquigarrow$	Then $X := G/K$ is an irreducible symmetric space of non-compact type
$SO^+(1, n)$		$\mathbb{R}H^n$

We wish to study the properties of semi-simple Lie groups (and its subgroups)/symmetric spaces of non-compact type (and its locally symmetric quotients).

Semi-simple Lie groups of non-compact type		**(Globally) symmetric spaces of non-compact type**
Then $G := \mathrm{Isom}(X)$ is a non-compact (almost) simple Lie group	$\leftsquigarrow$	Let X be an irreducible symmetric space of non-compact type
Let G be an (almost) simple Lie group and $K \leqslant G$ be the maximal compact subgroup	$\rightsquigarrow$	Then $X := G/K$ is an irreducible symmetric space of non-compact type
$SO^+(1,n)$		$\mathbb{R}H^n$

And we wish to use Lie theoretic facts about G to prove geometric properties of X.

$GL(n, \mathbb{R})$

$P(n, \mathbb{R})$

$P(n, \mathbb{R})$ is the set of all symmetric and positive definite $n \times n$ matrices, which is an open set in the vector space of all symmetric matrices $S(n, \mathbb{R})$.

We identify tangent bundle of $P(n, \mathbb{R})$ with $P(n, \mathbb{R}) \times S(n, \mathbb{R})$. Now the family of inner products

$$\langle X, Y \rangle_p := \mathrm{tr}(p^{-1} X p^{-1} Y)$$

for $p \in P(n, \mathbb{R})$ defines a Riemannian metric on $P(n, \mathbb{R})$.

Theorem

$P(n, \mathbb{R})$ is a CAT(0) symmetric space.

$GL(n, \mathbb{R})$ | $P(n, \mathbb{R})$

Theorem

$P(n, \mathbb{R})$ is a CAT(0) symmetric space.

Theorem

The action

$$t : GL(n, \mathbb{R}) \curvearrowright P(n, \mathbb{R})$$
$$t_g(p) := gpg^{\mathsf{T}}$$

is by isometries where the stabilizer of $I \in P(n, \mathbb{R})$ is $O(n, \mathbb{R})$.

$GL(n, \mathbb{R})$

$GL(n, \mathbb{R})$ has an order 2 automorphism

$$\begin{aligned} GL(n, \mathbb{R}) &\to GL(n, \mathbb{R}) \\ g &\mapsto (g^{-1})^{\mathsf{T}} \end{aligned}$$

which induces

$$\begin{aligned} \mathfrak{gl}(n, \mathbb{R}) &\to \mathfrak{gl}(n, \mathbb{R}) \\ X &\mapsto -X^{\mathsf{T}} \end{aligned}$$

This is a linear involution thus it is diagonalizable

$$\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{so}(n, \mathbb{R}) + S(n, \mathbb{R})$$

$P(n, \mathbb{R})$

Theorem

$P(n, \mathbb{R})$ is a CAT(0) symmetric space.

Theorem

The action

$$\begin{aligned} t : GL(n, \mathbb{R}) &\curvearrowright P(n, \mathbb{R}) \\ t_g(p) &:= gpg^{\mathsf{T}} \end{aligned}$$

is by isometries where the stabilizer of $I \in P(n, \mathbb{R})$ is $O(n, \mathbb{R})$.

T-closed, $\exp$-closed, closed subgroups of $GL(n,\mathbb{R})$ "completely" geodesic, closed and embedded submanifolds of $P(n,\mathbb{R})$

Definition

Let $G \leqslant GL(n,\mathbb{R})$ be a closed subgroup.

- G is called T-**closed** if it is closed under matrix transpose $G^{\mathsf{T}} = G$.
- G is called $\exp$-closed if $X \in S(n,\mathbb{R}), \exp(X) \in G$ implies $\exp(tX) \in G$ for all $t \in \mathbb{R}$

T-closed, $\exp$-closed, closed subgroups of $GL(n,\mathbb{R})$

Definition

Let $G \leqslant GL(n,\mathbb{R})$ be a closed subgroup.

- G is called **T-closed** if it is closed under matrix transpose $G^{\mathsf{T}} = G$.
- G is called $\exp$-closed if $X \in S(n,\mathbb{R}), \exp(X) \in G$ implies $\exp(tX) \in G$ for all $t \in \mathbb{R}$

$\rightsquigarrow$

"completely" geodesic, closed and embedded submanifolds of $P(n,\mathbb{R})$

Theorem

Let G be a T-closed, $\exp$-closed, closed subgroup of $GL(n,\mathbb{R})$. Then $X := G\{I\} = G \cap P(n,\mathbb{R})$ is a completely geodesic, closed and embedded submanifold. This implies it is a CAT(0) symmetric space.

T-closed, $\exp$-closed, closed subgroups of $GL(n, \mathbb{R})$

"completely" geodesic, closed and embedded submanifolds of $P(n, \mathbb{R})$

Let X be a completely geodesic, closed and embedded submanifold of $P(n, \mathbb{R})$. Then it is a CAT(0) symmetric space.

T-closed, $\exp$-closed, closed subgroups of $GL(n, \mathbb{R})$		"completely" geodesic, closed and embedded submanifolds of $P(n, \mathbb{R})$
Theorem *Let X be a completely geodesic, closed and embedded submanifold of $P(n, \mathbb{R})$. Then* $$G := \{g \in GL(n, \mathbb{R}) \vert t_g(X) = X\}$$ *is a closed,* T*-closed,* $\exp$*-closed subgroup of $GL(n, \mathbb{R})$ such that $X = G\{I\}$.*	$\leftsquigarrow$	Let X be a completely geodesic, closed and embedded submanifold of $P(n, \mathbb{R})$. Then it is a CAT(0) symmetric space.

Abelian subspaces of $\mathfrak{p}$	Flats in X

Let $\mathrm{Lie}(G) =: \mathfrak{g}$. The linear involution

$$\begin{aligned}\theta : \mathfrak{g} &\to \mathfrak{g}\\ X &\mapsto -X^{\mathsf{T}}\end{aligned}$$

is diagonalizable, thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

Abelian subspaces of $\mathfrak{p}$ — Flats in X

Let $\mathrm{Lie}(G) =: \mathfrak{g}$. The linear involution

$$\begin{aligned} \theta : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto -X^{\mathsf{T}} \end{aligned}$$

is diagonalizable, thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

$$\{\mathfrak{b} \subseteq \mathfrak{p} | \forall X, Y \in \mathfrak{b}, [X, Y] = 0\}$$

Abelian subspaces of $\mathfrak{p}$

Let $\mathrm{Lie}(G) =: \mathfrak{g}$. The linear involution

$$\begin{aligned}\theta : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto -X^{\mathsf{T}}\end{aligned}$$

is diagonalizable, thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

Flats in X

Definition

An embedded submanifold $F \subseteq X$ is a **flat** if it is distance isometric to $\mathbb{R}^n$.

We have a one-to one correspondence

$$\{\mathfrak{b} \subseteq \mathfrak{p} | \forall X, Y \in \mathfrak{b}, [X, Y] = 0\} \quad \leftrightarrow \quad \{\text{flat } F \subseteq X | I \in F\}$$

Abelian subspaces of $\mathfrak{p}$

Let $\mathrm{Lie}(G) =: \mathfrak{g}$. The linear involution

$$\begin{aligned} \theta : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto -X^{\mathsf{T}} \end{aligned}$$

is diagonalizable, thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

We have a one-to one correspondence

$$\{\mathfrak{b} \subseteq \mathfrak{p} | \forall X, Y \in \mathfrak{b}, [X, Y] = 0\} \quad \leftrightarrow \quad \{\text{flat } F \subseteq X | I \in F\}$$

Flats in X

Definition

An embedded submanifold $F \subseteq X$ is a **flat** if it is distance isometric to $\mathbb{R}^n$.

Definition

The **rank** of X is the dimension of one of its maximal flat.

Boundary of X

Definition

The boundary ∂X of a CAT(0) complete Riemannian manifold X is the set

$$\frac{\{\gamma : [0,\infty) \to X \text{ is a geodesic ray}\}}{\gamma_1 \sim \gamma_2 \iff \exists C > 0 : \forall t \geqslant 0, d(\gamma_1(t), \gamma_2(t)) \leqslant C}$$

Boundary of X

Definition

The boundary ∂X of a CAT(0) complete Riemannian manifold X is the set

$$\frac{\{\gamma : [0,\infty) \to X \text{ is a geodesic ray}\}}{\gamma_1 \sim \gamma_2 \iff \exists C > 0 : \forall t \geqslant 0, d(\gamma_1(t), \gamma_2(t)) \leqslant C}$$

We equip $\overline{X} := X \cup \partial X$ with the cone topology. Then the isometries of X extend as homeomorphisms on $\overline{X}$.

Boundary of X

$G_\xi := \{g \in G | g(\xi) = \xi\}$

Definition

The boundary ∂X of a CAT(0) complete Riemannian manifold X is the set

$$\frac{\{\gamma : [0,\infty) \to X \text{ is a geodesic ray}\}}{\gamma_1 \sim \gamma_2 \iff \exists C > 0 : \forall t \geqslant 0, d(\gamma_1(t), \gamma_2(t)) \leqslant C}$$

We equip $\overline{X} := X \cup \partial X$ with the cone topology. Then the isometries of X extend as homeomorphisms on $\overline{X}$.

Now we focus on the case of real hyperbolic space.

$O^+(1, n)$	$\mathbb{R}H^n$

Now we focus on the case of real hyperbolic space.

$O^+(1, n)$	$\mathbb{R}H^n$
	$\mathbb{R}H^n$ is an isotropic Riemannian manifold and a CAT(-1) metric space.

Now we focus on the case of real hyperbolic space.

$O^+(1,n)$	$\mathbb{R}H^n$
	$\mathbb{R}H^n$ is an isotropic Riemannian manifold and a CAT(-1) metric space. The rank of $\mathbb{R}H^n$ is 1.

$\Gamma \leqslant O^+(1,n)$ discrete $\qquad \Gamma \curvearrowright \overline{\mathbb{R}H^n}$

Consider a discrete subgroup Γ of the isometry group $\mathrm{Isom}(\mathbb{R}H^n) \cong O^+(1,n)$.

$\Gamma \leqslant O^+(1,n)$ discrete

Consider a discrete subgroup Γ of the isometry group $\mathrm{Isom}(\mathbb{R}H^n) \cong O^+(1,n)$.

$\Gamma \curvearrowright \overline{\mathbb{R}H^n}$

Definition

The **limit set** of Γ is the set

$$\Lambda_\Gamma := \overline{\Gamma\{x\}} \cap \partial\mathbb{R}H^n$$

for some $x \in \mathbb{R}H^n$.

Definition

The **critical exponent** $\delta_\Gamma \in [0, n-1]$ of Γ is the number

$$\inf\left\{\alpha > 0 \middle| \sum_{\gamma\in\Gamma} \exp(-\alpha d(\gamma(x), x)) < \infty\right\}$$

for some $x \in \mathbb{R}H^n$.

There is a proof of Mostow rigidity of real hyperbolic manifolds by constructing a measure supported on the limit set Λ_Γ (Patterson-Sullivan measure) and an ergodic measure for the geodesic flow on real hyperbolic manifolds (Bowen-Margulis measure).

There is a proof of Mostow rigidity of real hyperbolic manifolds by constructing a measure supported on the limit set Λ_Γ (Patterson-Sullivan measure) and an ergodic measure for the geodesic flow on real hyperbolic manifolds (Bowen-Margulis measure).

We hope to understand the proof and look at possible generalizations of the theory in higher rank.

There is a proof of Mostow rigidity of real hyperbolic manifolds by constructing a measure supported on the limit set Λ_Γ (Patterson-Sullivan measure) and an ergodic measure for the geodesic flow on real hyperbolic manifolds (Bowen-Margulis measure).

We hope to understand the proof and look at possible generalizations of the theory in higher rank.

Thank you!

References

1. Martin R. Bridson, Andre Haefliger - Metric Spaces of Non-Positive Curvature
2. Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature (1985)
3. Peter J Nicholls - The Ergodic Theory of Discrete Groups (1989)